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On the formulation of anisotropic elastic degradation. I. Theory based on a pseudo-logarithmic damage tensor rate

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Abstract

In spite of its appeal, anisotropic damage is being introduced in the constitutive equations of engineering materials at a slow pace. One of the main reasons is the difficulty of establishing general evolution laws. This originates from the lack of physical meaning of the thermodynamic forces conjugate to the damage variables, which finally constitute the space in which loading functions and 'damage rules' are defined. In this article, the authors propose a new 'pseudo-logarithmic' rate of damage, which has the advantage of exhibiting a simple and meaningful conjugate force with very convenient properties. A main advantage is the physical interpretation of the corresponding "damage rule", which clearly separates the effects of its volumetric part, responsible for isotropic degradation, from its deviatoric part, responsible for anisotropic effects. This new concept is applied to a second-order tensor secant formulation, which is developed using traditional concepts of continuum damage mechanics within the general theoretical framework of elastic degradation and damage recently proposed by the authors. A first example of anisotropic damage formulation based on these concepts, the 'generalized pseudo-Rankine' model, is presented and verified with analytical and numerical examples in a companion 'Part II' paper. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Since the introduction of the damage concept (Kachanov, 1958), models involving elastic degradation and damage have become progressively more popular and are nowadays often used for the constitutive description of quasi-brittle materials, such as concrete, rocks, ceramics, etc. (Dougill, 1976; Hueckel and Maier, 1977; Maier and Hueckel, 1979; Dragon and Mróz, 1979; Cordebois and Sidoroff, 1982; Ladevèze, 1983; Mazars and Lemaitre, 1984; Han and Chen, 1986; Ortiz, 1985; Simó and Ju, 1987; Chow and Wang, 1987a; Yazdani and Schreyer, 1988; Ju, 1989; Mazars and Pijaudier-Cabot, 1989; Chaboche, 1990; La

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Borderie et al., 1990; Chaboche, 1993; Lubarda et al., 1994; Carol and Willam, 1996). For a realistic description of the material behavior, many of these models incorporate additional features, such as different behavior in tension and compression, stiffness recovery due to microcrack closure, combination of degradation and plasticity, etc. From the theoretical viewpoint, however, all these features are not central to the 'core' description of elastic degradation and damage itself, which is the subject of the present study.

A detailed examination of the existing formulations of elastic degradation shows the need for further theoretical developments. On the one hand, terminology and notation are quite different from one author to another, making comparison time-consuming and often unclear. On the other hand, once the different formulations have been compared, it becomes apparent that significant theoretical gaps still exist, especially when anisotropic degradation is considered. In the recent past, these considerations motivated the authors to propose a unified theoretical framework for elastic degradation and damage (Carol et al., 1994). This proposal was based on well-known concepts and notations used traditionally in engineering elasto-plasticity, and led to very similar expressions for tangential stiffness and other ingredients of the constitutive theory. The benefits are significant: standardization provides a common language that is fundamental to real progress in any field; having similar expressions it is possible to take advantage of theoretical developments in elasto-plasticity, such as for instance closed-form solutions of strain localization analysis (Rizzi et al., 1995, 1996; Carol and Willam, 1997); moreover, combining damage and plasticity becomes equivalent to the formulation of multi-surface plasticity.

The theoretical framework of Carol et al. (1994) was developed in all generality with regard to the number and nature of the damage variables, and is briefly summarized in Section 2. In that publication, the authors have specified the general theory for the simplest type of elastic degradation: the "$(1-D)$" scalar damage formulation. In the present article, the possibilities of that constitutive framework are developed further by focusing on the next level of complexity, i.e. anisotropic elastic degradation of initially isotropic materials, based on second-order damage tensors. Within this topic, new efforts are directed towards three objectives: first to establish a 'basic' secant anisotropic formulation, which emerges naturally from the general theory; second (and the most important), to provide a simple and effective way to define appropriate and consistent evolution laws; and third, to implement a simple model based on these unifying concepts and present some numerical results.

The 'basic' secant formulation encompasses the simplest forms of isotropic and anisotropic damage descriptions that emerge most often in the existing literature, and at the same time appear to be theoretically consistent and compatible to each other (i.e., the isotropic is a particular case of the anisotropic formulation). Most scalar damage models in the literature (Leckie, 1978; Lemaitre and Chaboche, 1978; Mazars and Lemaitre, 1984; Resende, 1987; Simó and Ju, 1987; Ju, 1989; Franziskonis and Desai, 1987; Neilsen and Schreyer, 1992) are of the traditional "$(1-D)$" type, which in this article is called *'basic' isotropic formulation* and is described in Section 3. For anisotropic degradation, a number of models have been proposed, using damage vectors initially (Davidson and Stevens, 1973; Krajcinovic and Fonseka, 1981; Suaris and Shah, 1984; Costin, 1985), most often second-order tensors (Vakulenko and Kachanov, 1971; Dragon and Mróz, 1979; Kachanov, 1980; Murakami and Ohno, 1980; Betten, 1983; Oda, 1983; Chow and Wang, 1987b; Murakami, 1987; Suaris, 1987; Shen et al., 1989; Valanis, 1990; Hansen and Schreyer, 1992; Swoboda and Ito, 1993; Voyiadjis and Park, 1997), and even fourth- or eighth-order tensors (Chaboche, 1978, 1981; Lemaitre and Chaboche, 1990; Ortiz, 1985; Simó and Ju, 1987; Yazdani and Schreyer, 1988; Lubarda and Krajcinovic, 1993). The simplest representation of anisotropic degradation, which seems to be common to many proposals, and which satisfies at the same time basic consistency requirements, is based on the second-order damage tensor D_{ij} (direct generalization of the scalar D), or alternatively on the equivalent integrity tensor $\bar{\phi}_{ij} = \delta_{ij} - D_{ij}$ (Valanis, 1990), where δ_{ij} is the Kronecker delta. Although it has not been described in general in this way, the *'basic' anisotropic formulation* presented in Section 4 is derived using traditional concepts in continuum damage mechanics, such as effective stress, effective strain and energy equivalence, with resulting secant relations that are equivalent to those found in the literature.

It is not difficult to verify that the 'basic' formulation of Sections 3 and 4 represent a restricted form of damage in the isotropic and anisotropic ranges, which does not include some relatively simple forms of elastic degradation such as the 'von Mises' type (Ju, 1990; Neilsen and Schreyer, 1992). For the sake of clarity, however, attention in this article is focused on developing evolution laws and illustrative examples, and this is done on the 'basic' formulation. An 'extended' formulation with more general forms of both isotropic and anisotropic secant relations, which are still compatible with the general approach proposed, is developed in a separate article (Carol et al., 1999).

What can be considered to be well established in anisotropic degradation is often limited to the secant relations of stiffness or compliance in terms of the damage variables. In analogy to plasticity, a closed constitutive formulation also requires a loading function, a 'damage rule' and some hardening/softening laws. These are normally defined in the space of the thermodynamic 'forces' conjugate to the damage variables. With the usual second-order damage or integrity tensor, the conjugate forces turn out to be a product of stress, strain and damage tensor components, combined with the elastic constants, which does not exhibit a clear physical meaning (Valanis, 1990; Hansen and Schreyer, 1992). This makes it difficult to foresee the consequences of any specific choice of loading surface, damage rule, etc. As a remedy, the authors recently proposed to consider a new *pseudo-log rate of the damage tensor* (Carol et al., 1998), which exhibits great practical advantages, such as a very simple, physically meaningful conjugate force and a number of interesting properties concerning its volumetric–deviatoric decomposition and invariants. These new ideas are presented and are further developed in detail in Sections 5 and 6.

Finally, Section 7 includes some concluding remarks, which summarize the new theoretical developments presented. Application of these developments to the formulation of a new, specific, model for anisotropic damage and its analytical and numerical verification is the topic of a companion Part II article (Carol et al., 2000).

2. Theoretical framework for elastic degradation and damage

A general theoretical framework to formulate elastic degradation and damage at small strains was presented by the authors in Carol et al. (1994). This framework is briefly summarized in this section.

2.1. 'Plasticity format' of elastic degradation

The most characteristic equation of elastic degradation is the secant stress–strain relation:

$$\sigma_{ij} = E_{ijkl}\epsilon_{kl} \quad \text{or} \quad \epsilon_{ij} = C_{ijkl}\sigma_{kl}. \tag{1a, b}$$

E_{ijkl} and C_{ijkl} denote the components of the elastic secant fourth-order stiffness and compliance tensors **E** and **C**, which are assumed constant during unloading and reloading, and must remain symmetric to avoid spurious energy dissipation or generation under closed stress or strain paths in that range of behavior. Stiffness and compliance tensors are inverse to each other, i.e.,

$$E_{ijpq}C_{pqkl} = C_{ijpq}E_{pqkl} = I^{\text{sym}}_{ijkl}, \qquad I^{\text{sym}}_{ijkl} = \tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{2a, b}$$

Analogous to the concept of a plastic threshold condition, a loading function F is introduced to define an elastic domain in stress space $F(\boldsymbol{\sigma}, \mathbf{p}) < 0$ in which stiffness remains constant. Here, $\mathbf{p}$ denotes the set of variables that defines the current configuration of the elastic domain. Once the loading surface $F = 0$ is reached, further degradation may take place, $\dot{E}_{ijkl} \neq 0$, accompanied by increments of *degrading strain* $\dot{\epsilon}^{\text{d}}_{ij}$. The degrading strain rate is defined as the excess strain rate beyond the value that corresponds to the increment of stress according to the current secant stiffness (Fig. 1). With these definitions, the following set of rate equations describes progressive elastic degradation:

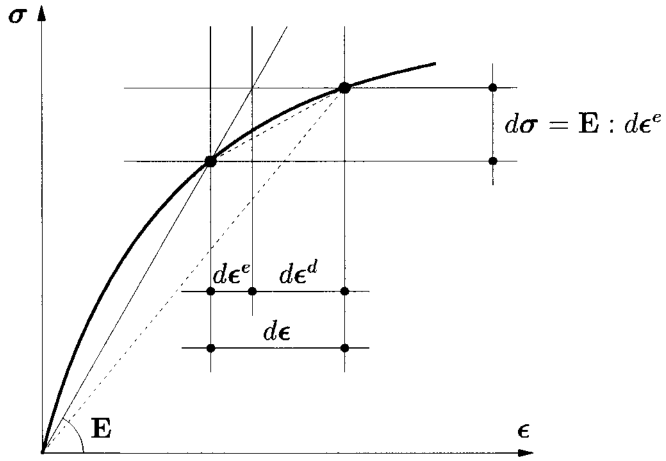

Fig. 1. Elastic and degrading strain increments.

$$\dot{\sigma}_{ij} = E_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}^{\mathrm{d}}_{kl}), \tag{3}$$

$$\dot{\epsilon}^{\mathrm{d}}_{kl} = \dot{\lambda}\ m_{kl} \quad \left(\text{optionally } m_{kl} = \frac{\partial Q}{\partial \sigma_{kl}}\right), \tag{4a, b}$$

$$\dot{F} = n_{ij}\dot{\sigma}_{ij} - H\dot{\lambda} = 0; \qquad n_{ij} = \left.\frac{\partial F}{\partial \sigma_{ij}}\right|_{\lambda}; \qquad H = -\left.\frac{\partial F}{\partial \lambda}\right|_{\sigma}, \tag{5a, b, c}$$

where **m** defines the direction of the degrading strain rates (4a), optionally expressed in terms of the gradient of a potential $Q(\boldsymbol{\sigma})$, (4b), and $\dot{\lambda}$ is the inelastic multiplier analogous to the plastic multiplier in traditional elasto-plasticity. Eq. (5a) represents the consistency condition, and H is the hardening/softening modulus.

Eqs. (3)–(5) are only valid for *further loading*, i.e. when $F = 0$, $\dot{F} = 0$ and $\dot{\lambda} \geqslant 0$. The alternative case is unloading, with $\dot{F} < 0$ and $\dot{\lambda} = 0$, in which only Eqs. (3) and (4) would be valid leading to $\dot{\epsilon}^{\mathrm{d}}_{kl} = 0$ and $\dot{\sigma}_{ij} = E_{ijkl}\dot{\epsilon}_{kl}$. Assuming that we are *on* the loading surface $F = 0$, the two cases may be distinguished using the complementarity (Kuhn–Tucker) conditions:

$$\dot{F} \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{F}\dot{\lambda} = 0. \tag{6a, b, c}$$

Eqs. (3)–(5) may be combined in the traditional way of plasticity, yielding the following expressions for the degradation multiplier and the tangent stiffness:

$$\dot{\lambda} = \frac{1}{\bar{H}}\ n_{cd}E_{cdkl}\dot{\epsilon}_{kl}, \qquad \bar{H} = H + n_{pq}E_{pqrs}m_{rs}, \tag{7a, b}$$

$$\dot{\sigma}_{ij} = E^{\tan}_{ijkl}\dot{\epsilon}_{kl}, \qquad E^{\tan}_{ijkl} = E_{ijkl} - \frac{1}{\bar{H}}E_{ijab}m_{ab}n_{cd}E_{cdkl}. \tag{8a, b}$$

Similarly to plasticity, the definitions of F and **m** are subjected to the constraint $\bar{H} > 0$ (i.e. $H > -\mathbf{n} : \mathbf{E} : \mathbf{m}$) such that the denominator in Eqs. (7a) and (8b) remains strictly positive. The model is called associated *in the stress space* when **m** is proportional to **n**, and consequently, the tangent stiffness exhibits major symmetry. If **m** is derived from a potential Q, associativity may be alternatively stated as $Q = F$.

2.2. Degradation rule for compliance

Previous Eqs. (3)–(8) have the same form as classical plasticity except for the secant stiffness instead of the initial stiffness, and the degrading strain instead of the plastic strain. In contrast to plasticity, however, those equations (and the additional definitions inherent to H) are not sufficient to define the evolution of an elastic-degrading model, since no evolution law has been specified for the (variable) secant stiffness or compliance. In order to do that, Eq. (1) can be differentiated and compared to Eq. (3). Using $\dot{E}_{ijkl} = -E_{ijpq}\dot{C}_{pqrs}E_{rskl}$ (obtained from differentiation of Eq. (2a)), this leads to

$$E_{ijpq}\dot{\epsilon}^{\mathrm{d}}_{pq} = -\dot{E}_{ijkl}\epsilon_{kl} \quad \text{or} \quad \dot{\epsilon}^{\mathrm{d}}_{pq} = \dot{C}_{pqrs}\sigma_{rs}, \tag{9a, b}$$

which relates the secant compliance and degrading strain rates. When the first is known, the second follows (but not the opposite). A generalized flow or *degradation rule* for the secant compliance can be formulated to define the evolution (growth) of degrading strains

$$\dot{C}_{ijkl} = \dot{\lambda}M_{ijkl} \quad \text{and} \quad m_{ij} = M_{ijkl}\sigma_{kl}, \tag{10a, b}$$

where $\dot{\lambda}$ specifies the magnitude and $\mathbf{M}$ the direction of the rate of change of $\mathbf{C}$. Eq. (10b) follows from replacing Eqs. (4a) and (10a) into Eq. (9b). This growth equation (essentially equivalent to Eq. (2.7) in Hueckel and Maier (1977) and to Eq. (3.36) in Ortiz (1985)) indicates that once the degradation rule has been established, the corresponding flow rule for degrading strains follows automatically. The requirement that $\mathbf{E}$ and $\mathbf{C}$ remain symmetric restricts $\mathbf{M}$ to be symmetric. With the specification of the degradation rule, the elastic-degrading formulation is closed. The final set of constitutive equations strictly required to integrate the material response for a prescribed strain history reduces to Eqs. (7), (10a) and (1) (and some appropriate hardening/softening laws inherent to H). In addition, the tangential stiffness given by Eq. (8) with $\mathbf{m}$ defined in Eq. (10b), may be needed for the incremental–iterative procedures, or in the analysis of strain localization properties based on the acoustic tensor (Rizzi et al., 1995, 1996; Carol and Willam, 1997).

2.3. Conjugate forces and associativity

Simple thermodynamic concepts lead to the definition of a fourth-order "force" tensor, conjugate to the increments of compliance in a similar way as stress is the conjugate variable to strain. For a constant temperature and disregarding effects other than mechanical, the energy stored per unit volume u may be assimilated to the energy that would be recovered upon unloading, which, at any given time, may be expressed in terms of current secant stiffness or compliance:

$$u = \tfrac{1}{2}\epsilon_{ij}E_{ijkl}\epsilon_{kl} = \tfrac{1}{2}\sigma_{ij}C_{ijkl}\sigma_{kl}. \tag{11a, b}$$

By differentiation, we obtain the balance of energies exchanged in a unit volume during a time increment dt (first principle):

$$\dot{u} = \sigma_{ij}\dot{\epsilon}_{ij} + \tfrac{1}{2}\epsilon_{ij}\dot{E}_{ijkl}\epsilon_{kl}. \tag{12}$$

In this equation, it is possible to identify $\sigma_{ij}\dot{\epsilon}_{ij}$ as the external work supply and $\dot{u}$ as the increase of elastic energy. Therefore, it is immediate to define the dissipation rate as the difference, which must remain non-negative (second principle):

$$\dot{d} = \sigma_{ij}\dot{\epsilon}_{ij} - \dot{u} = -\tfrac{1}{2}\epsilon_{ij}\dot{E}_{ijkl}\epsilon_{kl} = \tfrac{1}{2}\sigma_{ij}\dot{C}_{ijkl}\sigma_{kl} \geqslant 0. \tag{13}$$

The conjugate force $-\mathbf{Y}$ is then identified as

$$\dot{d} = (-Y_{ijkl})\dot{C}_{ijkl}, \qquad -Y_{ijkl} = \tfrac{1}{2}\sigma_{ij}\sigma_{kl}. \tag{14a, b}$$

With $-\mathbf{Y}$ it is possible to define the gradient of F in the compliance space and relate it to the gradient in stress space:

$$N_{ijkl} = \frac{\partial F}{\partial(-Y_{ijkl})}, \qquad n_{ij} = N_{ijkl}\sigma_{kl}. \tag{15a, b}$$

The concept of associativity can also be introduced in the compliance space, when **M** and **N** are parallel. Associativity in compliance space implies associativity in the stress space but not the opposite (Carol et al., 1994).

2.4. Elastic-damage formulation

In the formulation described in Sections 2.1 and 2.2, the degradation state is characterized by the secant compliance (or stiffness) tensor itself, with 21 independent components. The corresponding evolution laws must also involve 21 components (those of the tensor M_{ijkl}). Alternatively, it is reasonable to assume a reduced set of variables, which still fully characterize the state of degradation or *damage* in the material for which simple evolution laws can be postulated. These are the *damage variables*, $\mathscr{D}_*$, the number and nature of which (scalar, vectorial or tensorial) does not need to be specified for the development of the general theory (the subscript $*$ represents the desired number of indices). According to that concept, one may write

$$C_{ijkl} = C_{ijkl}(C^0_{pqrs}, \mathscr{D}_*), \qquad \dot{C}_{ijkl} = \frac{\partial C_{ijkl}}{\partial \mathscr{D}_*}\dot{\mathscr{D}}_*, \tag{16a, b}$$

where C^0_{ijkl} is the initial compliance, C_{ijkl} are a set of known, continuous and differentiable functions, and repetition of subscript $*$ implies summation over all the indices represented by the symbol. A damage rule for $\mathscr{D}_*$ may be formulated, and its relation to the evolution rule for compliance may be established as

$$\dot{\mathscr{D}}_* = \dot{\lambda}\mathscr{M}_*, \qquad M_{ijkl} = \frac{\partial C_{ijkl}}{\partial \mathscr{D}_*}\mathscr{M}_*. \tag{17a, b}$$

Similar to Eqs. (4a) and (10a), $\dot{\lambda}$ specifies the intensity and $\mathscr{M}_*$ the direction of the increment of the damage variables in the *damage space*. The final equations for the evolution of elastic damage are the same as for the elastic degradation where M_{ijkl} is replaced by $\mathscr{M}_*$.

Similar thermodynamic concepts as before lead to the conjugate force $-\mathscr{Y}_*$ to the damage variable $\dot{\mathscr{D}}_*$, which is

$$-\mathscr{Y}_* = -Y_{ijkl}\frac{\partial C_{ijkl}}{\partial \mathscr{D}_*} \tag{18}$$

and allows one to define the gradient of F in the damage space and its relation to **N**:

$$\mathscr{N}_* = \frac{\partial F}{\partial(-\mathscr{Y}_*)}, \qquad N_{ijkl} = \frac{\partial C_{ijkl}}{\partial \mathscr{D}_*}\mathscr{N}_*. \tag{19a, b}$$

Associativity in the damage space occurs when $\mathscr{N}_*$ is proportional to $\mathscr{M}_*$, and it also implies associativity at compliance and stress levels. Further details of this theoretical framework, as well as the dual formulation in strain space and the equivalences between both versions of the theory, may be found in Carol et al. (1994).

3. 'Basic' isotropic damage

Using the theoretical framework described in Section 2, it is possible to formulate a variety of damage models. The simplest ones are those in which the initial stiffness (and therefore also the compliance) is isotropic, and its degraded counterpart also maintains isotropy. In particular, the traditional "$(1-D)$" scalar damage model is that one in which all the components of the stiffness tensor are reduced with the same coefficient $(1-D)$, where D is a damage variable varying from 0 to 1. In Carol et al. (1994), a strain-based formulation of this type was derived in the general framework presented, and it was shown that a number of models available in the literature (Mazars and Lemaitre, 1984; Simó and Ju, 1987; Neilsen and Schreyer, 1992) were included as particular cases. Here, the derivation is presented in stress space and with more convenient choices of inelastic multiplier and damage variable, which makes expressions look simpler and allows us to introduce the concept of logarithmic scalar damage, although the resulting formulation is fully equivalent.

First, consider the general form of the isotropic stiffness and compliance tensors:

$$E_{ijkl} = \Lambda\delta_{ij}\delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \qquad C_{ijkl} = \frac{-\nu}{E}\delta_{ij}\delta_{kl} + \frac{1+\nu}{2E}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{20a, b}$$

where Λ and G are the Lamé constants, linked to Young's modulus E and Poisson's ratio ν by the classical relations

$$\Lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad G = \frac{E}{2(1+\nu)}. \tag{21a, b}$$

In the "$(1-D)$" scalar damage model, the following well-known expressions are assumed for the secant stiffness and its inverse compliance:

$$E_{ijkl} = (1-D)E^0_{ijkl}, \qquad C_{ijkl} = \frac{1}{1-D}C^0_{ijkl}, \tag{22a, b}$$

where E^0_{ijkl} and C^0_{ijkl} are the initial stiffness and compliance tensors given by Eq. (20a) with initial values of elastic constants Λ^0, G^0 or E^0, ν^0. i.e.

$$E^0_{ijkl} = \Lambda^0\delta_{ij}\delta_{kl} + G^0\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \qquad C^0_{ijkl} = \frac{-\nu^0}{E^0}\delta_{ij}\delta_{kl} + \frac{1+\nu^0}{2E^0}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{23a, b}$$

Differentiating Eq. (22b) yields

$$\dot{C}_{ijkl} = \frac{\dot{D}}{(1-D)^2}C^0_{ijkl}. \tag{24}$$

A new logarithmic scalar damage variable L is defined:

$$\mathscr{D}_* = \text{scalar} = L = \ln\frac{1}{1-D}, \qquad D = 1 - \mathrm{e}^{-L}, \tag{25a, b, c}$$

which allows us to rewrite Eq. (22) as

$$E_{ijkl} = \mathrm{e}^{-L}E^0_{ijkl}, \qquad C_{ijkl} = \mathrm{e}^{L}C^0_{ijkl}. \tag{26a, b}$$

While the conventional damage variable D varies between 0 and 1, the logarithmic damage L varies between 0 and ∞ as represented in Fig. 2a and b. Evolution of the secant Young's modulus with L, which is implicit in previous equations (i.e. $E = \mathrm{e}^{-L}E^0$), is represented in Fig. 2c.

Differentiating Eq. (26b), we can rewrite Eq. (24) as

$$\dot{C}_{ijkl} = \dot{L}\mathrm{e}^{L}C^0_{ijkl} = \dot{L}C_{ijkl}. \tag{27}$$

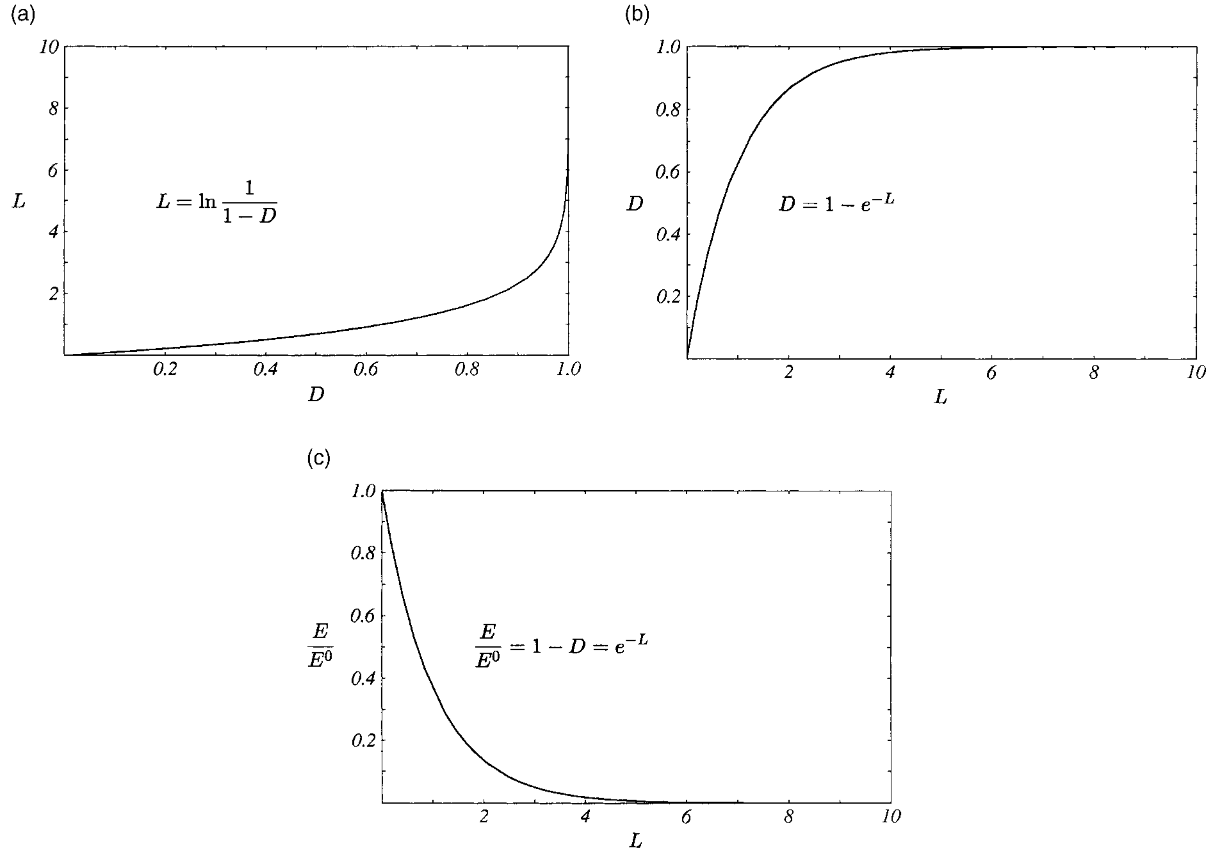

Fig. 2. Logarithmic scalar damage L: (a) relation to traditional scalar damage $L(D)$, (b) inverse relation $D(L)$, and (c) relation $E(L)/E^0$.

With previous definitions, it is possible to use $\dot{L}$ itself as inelastic multiplier:

$$\dot{\lambda} = \dot{L} = \frac{\dot{D}}{1-D}. \tag{28}$$

This leads to the identification of the m terms of the general theory, which take the convenient simple form of the current value of compliance and strain:

$$\frac{\partial C_{ijkl}}{\partial \mathscr{D}_*} = \frac{\partial C_{ijkl}}{\partial \lambda} = C_{ijkl}, \qquad \mathscr{M}_* = 1, \tag{29a, b}$$

$$M_{ijkl} = C_{ijkl}, \qquad m_{ij} = C_{ijkl}\sigma_{kl} = \epsilon_{ij}. \tag{29c, d}$$

The dissipation equation leads to the force $-\mathscr{Y}$ conjugate to the logarithmic damage L, which turns out to be equal to the current (secant) elastic energy:

$$\dot{d} = \tfrac{1}{2}\sigma_{ij}C_{ijkl}\sigma_{kl}\dot{L} = -\mathscr{Y}\dot{L}, \qquad -\mathscr{Y} = u. \tag{30a, b}$$

In order to achieve an associated formulation, the loading function is written in terms of the conjugate force $-\mathscr{Y} = u$ and the damage state L (equivalent to D), in the format

$$F = u - r(L) = 0. \tag{31}$$

This is a general form of $F(u,L)$ since, from any other expression $F'(u,L) = 0$, one can always isolate $u = r(L)$ and rewrite as above (in particular, this definition includes other functions usually found in the literature such as those written in terms of the stress- or strain-based undamaged energies $u^0 = \sigma_{ij} C^0_{ijkl} \sigma_{kl}/2 = (1-D)u$ or $\bar{u}^0 = \epsilon_{ij} E^0_{ijkl} \epsilon_{kl}/2 = u/(1-D)$).

From F, the various gradients of the loading function at constant λ may be obtained:

$$\mathscr{N} = \frac{\partial F}{\partial(-\mathscr{Y})} = 1, \qquad N_{ijkl} = \frac{\partial F}{\partial(-Y)_{ijkl}} = C_{ijkl}, \tag{32a, b}$$

$$n_{ij} = \frac{\partial F}{\partial \sigma_{ij}} = C_{ijkl}\sigma_{kl} = \epsilon_{ij}. \tag{33}$$

Note that the three gradients $\mathscr{N}$, **N** and **n** are equal to the corresponding rules $\mathscr{M}$, **M** and **m** in the theory, which means associativity at all levels. In general, associativity depends on the particular choice of F such that its gradients are parallel to the damage rule. In the case of scalar damage, however, because both damage rule and gradient of F are scalars, it is sufficient that $\mathscr{N}$ exists and it will automatically be parallel to $\mathscr{M}$. Therefore, the only condition for full associativity is that F be expressed in terms of the conjugate force, i.e. in this case, of u (a more detailed discussion on the various levels of associativity in damage models and related considerations may be found in Carol et al. (1994)).

The hardening/softening modulus $H = -\partial F/\partial\lambda$ at constant stress, is also obtained from Eq. (31) as

$$H = \frac{\partial r}{\partial L} - u. \tag{34}$$

Finally, with m_{ij}, n_{ij} and H, the expression for the tangent stiffness is obtained:

$$E^{\tan}_{ijkl} = \mathrm{e}^{-L}\ E^0_{ijkl} - \frac{1}{\bar{H}}\sigma_{ij}\sigma_{kl}, \qquad \bar{H} = \frac{\partial r}{\partial L} + u. \tag{35a, b}$$

As described, the model has only the hardening/softening function $r(L)$ (or, equivalently, $r(D)$) to be defined. This function may be identified from a single stress–strain curve from experiments, for instance from a uniaxial test. Once it has been chosen, however, all other features of the model are automatically fixed.

If further degrees of freedom are needed in the model in order to fit additional experimental data without abandoning the domain of isotropic degradation, the model would have to be modified. In order to focus on the main objective of evolution laws based on the pseudo-log damage rate, this will not be pursued in this paper. However, a simple extension along this line has already been advanced in (Carol et al., 1998) and is developed in more detail and inserted in the general context of an ‘extended’ anisotropic degradation in (Carol et al., 1999).

4. ‘Basic’ anisotropic damage using CDM concepts: secant relations

As explained in Section 2.4, the first step in formulating a specific damage model consists of defining the damage variables and the dependency of the secant stiffness or compliance (16a) on those variables. In the case of traditional “$(1-D)$” isotropic damage, this was trivial as given by expression (22a,b). For anisotropic degradation based on second-order tensor damage variables, however, the task becomes considerably more complicated. In the literature on continuum damage mechanics, additional concepts are often introduced such as effective stress and effective strain, as well as strain equivalence, stress equivalence and energy equivalence. With these concepts, it is possible to establish simplified physical micro–macro models of the degraded material, that are the basis to relate the anisotropic secant stiffness and compliance operators to the damage variables.

4.1. Effective stress and strain, energy equivalence

Degradation may be understood as the average effect of distributed microcracks. *Effective stress* $\sigma_{ij}^{\mathrm{eff}}$ and *effective strain* $\epsilon_{ij}^{\mathrm{eff}}$ are defined as stress and strain to which the material skeleton between microcracks is subjected. In this context, the relation between effective stress and effective strain describes the constitutive behavior of the undamaged material, which for the sake of simplicity is assumed to be linear elastic and isotropic:

$$\sigma_{ij}^{\mathrm{eff}} = E_{ijkl}^{0}\epsilon_{kl}^{\mathrm{eff}}, \qquad \epsilon_{ij}^{\mathrm{eff}} = C_{ijkl}^{0}\sigma_{kl}^{\mathrm{eff}}, \tag{36a, b}$$

where E_{ijkl}^{0} and C_{ijkl}^{0} are given by Eqs. (23a,b).

Henceforth, the damage variables must relate the effective quantities to their *nominal* or apparent counterparts, which are the ones that are measured externally and satisfy equilibrium and compatibility at structural level. In the literature, the relation between nominal and effective quantities has been established in three ways: strain equivalence, stress equivalence and energy equivalence. In analogy to composite mechanics, *strain equivalence* (Lemaitre and Chaboche, 1990) infers that effective and nominal strains are equal and stresses differ, while *stress equivalence* refers to the opposite. These assumptions may be interpreted microscopically in terms of parallel or serial arrangements of elements which fail during the degradation process according to the Voigt and the Reuss models. In spite of strain equivalence being widely used, this approach (and also the stress equivalence approach) exhibits the significant theoretical shortcoming of producing non-symmetric secant stiffness and compliance tensors, which introduces loss of energy conservation in the unloading–reloading regime.

In contrast, *energy equivalence* automatically induces symmetry in the secant stiffness and compliance tensors. In this approach (Cordebois and Sidoroff, 1982), the elastic energy stored in terms of effective quantities with undamaged stiffness and in terms of nominal quantities with secant stiffness must be the same (this definition actually requires the undamaged behavior to be linear elastic; a more general derivation without that requirement, which is based on the principle of virtual work, may be found in Carol and Bažant (1997)).

In the energy equivalence approach, neither effective strain nor effective stress coincide with their nominal counterparts. Rather, assuming that the nominal-effective relations are linear, they must be given by the same fourth-order "damage-effect" tensor $\bar{\alpha}_{ijkl}$ in the following form:

$$\sigma_{ij} = \bar{\alpha}_{ijkl}\sigma_{kl}^{\mathrm{eff}}, \qquad \epsilon_{ij}^{\mathrm{eff}} = \bar{\alpha}_{klij}\epsilon_{kl}, \tag{37a, b}$$

which can be also written in inverted form as:

$$\sigma_{ij}^{\mathrm{eff}} = \alpha_{ijkl}\sigma_{kl}, \qquad \epsilon_{ij} = \alpha_{klij}\epsilon_{kl}^{\mathrm{eff}}, \tag{38a, b}$$

where $\bar{\alpha}_{ijkl}$ and α_{ijkl} are tensors inverse to each other, with all minor (but not necessarily major) symmetries, i.e.

$$\alpha_{ijpq}\bar{\alpha}_{pqkl} = \bar{\alpha}_{ijpq}\alpha_{pqkl} = I_{ijkl}^{\mathrm{sym}}. \tag{39}$$

Combining Eqs. (37) and (38) with Eq. (36), one recovers the secant relations (1a,b), where

$$E_{ijkl} = \bar{\alpha}_{ijpq}E_{pqrs}^{0}\bar{\alpha}_{klrs}, \qquad C_{ijkl} = \alpha_{pqij}C_{pqrs}^{0}\alpha_{rskl}. \tag{40a, b}$$

Hereby, major symmetry of secant stiffness and compliance is indeed guaranteed in this approach. Notationwise, symbols with overbar are used in the stiffness version of the model and symbols without in the dual compliance-based counterpart (in agreement with the general theory, Section 2).

4.2. Isotropic damage in the CDM context

It is useful to rephrase the "$(1-D)$" isotropic damage model from Section 3 within the CDM environment just introduced. To do that, we assume

$$\bar{\alpha}_{ijkl} = \bar{\phi}\ I^{\mathrm{sym}}_{ijkl}, \qquad \alpha_{ijkl} = \phi\ I^{\mathrm{sym}}_{ijkl}. \tag{41a, b}$$

These tensors may be written in matrix notation as

$$\bar{\boldsymbol{\alpha}} = \bar{\phi} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}; \qquad \boldsymbol{\alpha} = \phi \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}. \tag{42a, b}$$

Note that, in this matrix representation, $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}$ relate two six-component vectors of the same nature, i.e. stresses $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^{\mathrm{T}}$ and $\boldsymbol{\sigma}^{\mathrm{eff}} = \left[\sigma^{\mathrm{eff}}_{11}, \sigma^{\mathrm{eff}}_{22}, \sigma^{\mathrm{eff}}_{33}, \sigma^{\mathrm{eff}}_{12}, \sigma^{\mathrm{eff}}_{23}, \sigma^{\mathrm{eff}}_{31}\right]^{\mathrm{T}}$, or strains $\boldsymbol{\epsilon} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{31}]^{\mathrm{T}}$ and $\boldsymbol{\epsilon}^{\mathrm{eff}} = \left[\epsilon^{\mathrm{eff}}_{11}, \epsilon^{\mathrm{eff}}_{22}, \epsilon^{\mathrm{eff}}_{33}, 2\epsilon^{\mathrm{eff}}_{12}, 2\epsilon^{\mathrm{eff}}_{23}, 2\epsilon^{\mathrm{eff}}_{31}\right]^{\mathrm{T}}$, and therefore, all the terms on the right half of the matrix actually correspond to the sum of two tensor components (e.g. the term 4,4 of the matrix $\boldsymbol{\alpha}$ corresponds to $\alpha_{1212} + \alpha_{1221}$, etc.).

Replacing Eq. (41) into Eqs. (37) and (38), the nominal-effective relations are obtained, which in this case exhibit simple proportionality:

$$\sigma_{ij} = \bar{\phi}\ \sigma^{\mathrm{eff}}_{ij}, \quad \sigma^{\mathrm{eff}}_{ij} = \phi\ \sigma_{ij}, \quad \epsilon^{\mathrm{eff}}_{ij} = \bar{\phi}\ \epsilon_{ij}, \quad \epsilon_{ij} = \phi\ \epsilon^{\mathrm{eff}}_{ij}. \tag{43a, b, c, d}$$

These equations may now be substituted into Eqs. (36a,b) and the result compared to Eq. (22), which leads to the relations between the scalar quantities $\bar{\phi}$ and ϕ, and the traditional scalar damage D:

$$\bar{\phi} = \frac{1}{\phi} = \sqrt{1-D}. \tag{44}$$

4.3. Second-order damage tensors

Disregarding vector-valued damage descriptions because of theoretical and practical shortcomings (Leckie and Onat, 1981; Carol et al., 1991), a second-order symmetric tensor seems to be the simplest option to represent anisotropic damage in a consistent manner. Similar to the stress or strain tensors, the second-order symmetric damage tensor can also be decomposed spectrally and represented graphically in a convenient way. All those advantages were recognized by several authors who proposed either the direct generalization of D to a *second-order symmetric damage tensor* D_{ij}, which varies between zero and δ_{ij} as damage progresses (Murakami and Ohno, 1980; Cordebois and Sidoroff, 1982; Murakami, 1987), or the use of an *integrity tensor* $\bar{\phi}_{ij} = \delta_{ij} - D_{ij}$, which has exactly the opposite variation (Betten, 1983; Valanis, 1990). The two tensors $\bar{\phi}_{ij}$ and D_{ij} share principal axes and their principal values vary between 0 and 1, and are related according to $D_{(i)} = 1 - \bar{\phi}_{(i)}$.

Actually, one may think of a number of second-order tensors to characterize damage, all with the same principal axes and simple relations between their principal values; the choice of which is mainly a matter of convenience. Additional to the integrity tensor $\bar{\phi}_{ij}$, we introduce its square root $\bar{w}_{ij}$ (which also varies from δ_{ij} to 0) and their inverses ϕ_{ij} and w_{ij} (which vary from δ_{ij} to ∞). These tensors are all symmetric, and their principal values satisfy the following relations:

$$\bar{\phi}_{ij} = \bar{w}_{ik}\bar{w}_{kj}, \qquad \bar{\phi}_{(i)} = \bar{w}^2_{(i)}, \tag{45a, b}$$

$$\phi_{ij} = w_{ik}w_{kj}, \qquad \phi_{(i)} = w^2_{(i)}, \tag{46a, b}$$

$$\bar{\phi}_{ik}\phi_{kj} = \phi_{ik}\bar{\phi}_{kj} = \delta_{ij}, \qquad \bar{\phi}_{(i)} = \frac{1}{\phi_{(i)}}, \tag{47a, b}$$

$$\bar{w}_{ik}w_{kj} = w_{ik}\bar{w}_{kj} = \delta_{ij}, \qquad \bar{w}_{(i)} = \frac{1}{w_{(i)}}. \tag{48a, b}$$

In the case of isotropic degradation, all these tensors reduce to their volumetric form:

$$\bar{\phi}_{ij} = \bar{\phi}\delta_{ij}, \quad \phi_{ij} = \phi\delta_{ij}, \quad \bar{w}_{ij} = \bar{w}\delta_{ij}, \quad w_{ij} = w\delta_{ij}. \tag{49a, b, c, d}$$

4.4. 'Basic' anisotropic formulation based on product-type symmetrization

After introducing the tensor-valued damage variables, the nominal-effective relations (37) are established so that the damage-effect tensors $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\alpha}}$ can be identified. However, attempting a direct generalization of the one-dimensional relation $\sigma = \bar{\phi}\sigma^{\text{eff}}$ where $\bar{\phi}$ represents an effective area reduction, one finds $\sigma_{ij} = \bar{\phi}_{ik}\sigma^{\text{eff}}_{kj}$, where symmetry cannot be ensured for σ_{ij} even if σ^{eff}_{kj} and $\bar{\phi}_{ik}$ are symmetric. This indicates that some form of symmetrization should be applied. In the literature, both *"sum-type"* and *"product-type"* symmetrizations have been considered (Voyiadjis and Park, 1997). Additionally, each of them can be developed either in terms of stress or in terms of strain, leading to various forms of the damage-effect tensors $\bar{\alpha}_{ijkl}$ or α_{ijkl}. However, a careful examination shows that, with 'product-type' symmetrization, both versions of the tensor-valued damage formulation (i.e. strain- and stress-based) are fully equivalent, whereas with the 'sum type' they are not. For this reason, product symmetrization is the one considered in the following. The parallel derivation with sum-type symmetrization is enclosed as Appendix B.

Product-type symmetrization of effective stresses was originally proposed (Cordebois and Sidoroff, 1982) as

$$\sigma_{ij} = \bar{w}_{ik}\sigma^{\text{eff}}_{kl}\bar{w}_{lj}. \tag{50}$$

This may be conveniently rewritten in terms of the damage-effect tensor $\bar{\alpha}_{ijkl}$:

$$\sigma_{ij} = \bar{\alpha}_{ijkl}\sigma^{\text{eff}}_{kl}, \qquad \bar{\alpha}_{ijkl} = \tfrac{1}{2}(\bar{w}_{ik}\bar{w}_{jl} + \bar{w}_{il}\bar{w}_{jk}), \tag{51a, b}$$

where advantage has been taken of the symmetry of $\boldsymbol{\sigma}^{\text{eff}}$ to obtain an $\bar{\boldsymbol{\alpha}}$ with all minor (and major) symmetries. Note that, due to the energy equivalence approach and related expressions (37b), this assumption also implies

$$\epsilon^{\text{eff}}_{ij} = \bar{w}_{ik}\epsilon_{kl}\bar{w}_{lj}. \tag{52}$$

Matrix representation of the tensor $\bar{\alpha}_{ijkl}$ in the principal axes of damage (i.e. the principal axes of D_{ij}, ϕ_{ij}, etc.) exhibits the following diagonal form, which has been used repeatedly in the literature on anisotropic damage (Shen et al., 1989; Chow and Wang, 1987b; Hansen and Schreyer, 1992):

$$\bar{\boldsymbol{\alpha}} = \begin{bmatrix} \bar{\phi}_{(1)} & & & & & \\ & \bar{\phi}_{(2)} & & & & \\ & & \bar{\phi}_{(3)} & & & \\ & & & \sqrt{\bar{\phi}_{(1)}\bar{\phi}_{(2)}} & & \\ & & & & \sqrt{\bar{\phi}_{(2)}\bar{\phi}_{(3)}} & \\ & & & & & \sqrt{\bar{\phi}_{(3)}\bar{\phi}_{(1)}} \end{bmatrix}. \tag{53}$$

Note that the term affecting the shear–stress component i, j is a product-type average of $\bar{\phi}_i$ and $\bar{\phi}_j$ instead of the sum-type average obtained using the 'sum-type' approach (Appendix B, Eq. (B.3)). To illustrate the difference, consider for instance the case in which damage is fully mobilized in principal direction 1, whereas it is zero in principal direction 2. In the summation approach, the damage effect coefficient $\bar{\alpha}_{1212} + \bar{\alpha}_{1221}$ would be 0.5 (i.e. the shear 'effective area' would still be half of the original), whereas in the product approach it is zero, i.e. no shear stress-carrying area remains.

Now, replacing Eq. (51b) in Eq. (40a) and taking advantage of the minor symmetries of $\mathbf{E}^0$, one obtains

$$E_{ijkl} = \bar{w}_{ip}\bar{w}_{jq}\bar{w}_{kr}\bar{w}_{ls}E^0_{pqrs}. \tag{54}$$

Further, replacing the initial elastic stiffness (23a), and making the appropriate products and substitutions, one obtains

$$E_{ijkl} = \Lambda^0\bar{\phi}_{ij}\bar{\phi}_{kl} + G^0\left(\bar{\phi}_{ik}\bar{\phi}_{jl} + \bar{\phi}_{il}\bar{\phi}_{jk}\right), \tag{55}$$

which can be rewritten in terms of any other pair of elastic constants, obtaining in each case expressions analogous to the initial isotropic stiffness tensor, in which all Kronecker deltas δ_{ij} have been replaced with $\bar{\phi}_{ij}$. Note that this expression for E_{ijkl} actually corresponds to the model initially proposed by Valanis (1990), although that derivation was made starting from a specific form of the elastic potential, without resorting to the concepts of effective stress, effective strain or energy equivalence.

In the principal axes of damage, the secant stiffness and compliance tensors may be written as 6×6 matrices:

$$\mathbf{E} = \begin{bmatrix} \bar{\phi}^2_{(1)}(\Lambda^0 + 2G^0) & \bar{\phi}_{(1)}\bar{\phi}_{(2)}\Lambda^0 & \bar{\phi}_{(1)}\bar{\phi}_{(3)}\Lambda^0 & & & \\ \bar{\phi}_{(2)}\bar{\phi}_{(1)}\Lambda^0 & \bar{\phi}^2_{(2)}(\Lambda^0 + 2G^0) & \bar{\phi}_{(2)}\bar{\phi}_{(3)}\Lambda^0 & & & \\ \bar{\phi}_{(3)}\bar{\phi}_{(1)}\Lambda^0 & \bar{\phi}_{(3)}\bar{\phi}_{(2)}\Lambda^0 & \bar{\phi}^2_{(3)}(\Lambda^0 + 2G^0) & & & \\ & & & \bar{\phi}_{(1)}\bar{\phi}_{(2)}G^0 & & \\ & & & & \bar{\phi}_{(2)}\bar{\phi}_{(3)}G^0 & \\ & & & & & \bar{\phi}_{(3)}\bar{\phi}_{(1)}G^0 \end{bmatrix}, \tag{56}$$

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\bar{\phi}^2_{(1)}E^0} & \frac{-\nu^0}{\bar{\phi}_{(1)}\bar{\phi}_{(2)}E^0} & \frac{-\nu^0}{\bar{\phi}_{(1)}\bar{\phi}_{(3)}E^0} & & & \\ \frac{-\nu^0}{\bar{\phi}_{(1)}\bar{\phi}_{(2)}E^0} & \frac{1}{\bar{\phi}^2_{(2)}E^0} & \frac{-\nu^0}{\bar{\phi}_{(2)}\bar{\phi}_{(3)}E^0} & & & \\ \frac{-\nu^0}{\bar{\phi}_{(1)}\bar{\phi}_{(3)}E^0} & \frac{-\nu^0}{\bar{\phi}_{(2)}\bar{\phi}_{(3)}E^0} & \frac{1}{\bar{\phi}^2_{(3)}E^0} & & & \\ & & & \frac{2(1+\nu^0)}{\bar{\phi}_{(1)}\bar{\phi}_{(2)}E^0} & & \\ & & & & \frac{2(1+\nu^0)}{\bar{\phi}_{(2)}\bar{\phi}_{(3)}E^0} & \\ & & & & & \frac{2(1+\nu^0)}{\bar{\phi}_{(1)}\bar{\phi}_{(3)}E^0} \end{bmatrix}. \tag{57}$$

Note that, in order to obtain work conjugates, vector representation of strains includes the usual factor 2 in the shear components, while stresses do not. For this reason, matrix representation of stiffness and compliance does not follow the same rules as that of damage-effect tensors used in a previous section. For **E**, terms on the right half of the matrix are equal to *one half* of the sum of the corresponding tensor components (i.e. the term 4,4 of **E** is equal to $(E_{1212} + E_{1221})/2 = E_{1212}$ due to minor symmetries); while for **C**, terms on the lower half of the matrix are equal to *twice* the sum of the corresponding tensor components (i.e. term 4,4 of the matrix is equal to $2(C_{1212} + C_{1221}) = 4C_{1212}$).

Compliance matrix (57) may be compared to the traditional compliance matrix of orthotropic elasticity:

$$\mathbf{C}^{\text{orth}} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_2} & \frac{-\nu_{13}}{E_3} & & & \\ \frac{-\nu_{21}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_3} & & & \\ \frac{-\nu_{31}}{E_1} & \frac{-\nu_{32}}{E_2} & \frac{1}{E_3} & & & \\ & & & \frac{1}{G_{12}} & & \\ & & & & \frac{1}{G_{23}} & \\ & & & & & \frac{1}{G_{31}} \end{bmatrix}, \tag{58}$$

which results in the following equivalences:

$$\begin{aligned} &E_1 = \bar{\phi}^2_{(1)}E^0, \quad E_2 = \bar{\phi}^2_{(2)}E^0, \quad E_3 = \bar{\phi}^2_{(3)}E^0, \\ &\nu_{12} = \frac{\bar{\phi}_{(2)}}{\bar{\phi}_{(1)}}\nu^0, \quad \nu_{13} = \frac{\bar{\phi}_{(3)}}{\bar{\phi}_{(1)}}\nu^0, \quad \nu_{21} = \frac{\bar{\phi}_{(1)}}{\bar{\phi}_{(2)}}\nu^0, \\ &\nu_{23} = \frac{\bar{\phi}_{(3)}}{\bar{\phi}_{(2)}}\nu^0, \quad \nu_{31} = \frac{\bar{\phi}_{(1)}}{\bar{\phi}_{(3)}}\nu^0, \quad \nu_{32} = \frac{\bar{\phi}_{(2)}}{\bar{\phi}_{(3)}}\nu^0, \\ &G_{12} = \bar{\phi}_{(1)}\bar{\phi}_{(2)}\frac{E^0}{2(1+\nu^0)}, \quad G_{23} = \bar{\phi}_{(2)}\bar{\phi}_{(3)}\frac{E^0}{2(1+\nu^0)}, \quad G_{31} = \bar{\phi}_{(3)}\bar{\phi}_{(1)}\frac{E^0}{2(1+\nu^0)}. \end{aligned} \tag{59a–l}$$

The dual formulation of the product-type, in terms of strain/compliance, with its dual damage-effect tensor α_{ijkl}, secant compliance C_{ijkl} and their matrix representations, yields

$$\epsilon_{ij} = w_{ik}\epsilon^{\text{eff}}_{kl}w_{lj}, \qquad \sigma^{\text{eff}}_{ij} = w_{ik}\sigma_{kl}w_{lj}, \tag{60a, b}$$

$$\epsilon_{ij} = \alpha_{ijkl}\epsilon^{\text{eff}}_{kl}, \qquad \alpha_{ijkl} = \tfrac{1}{2}(w_{ik}w_{jl} + w_{il}w_{jk}), \tag{61a, b}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \phi_{(1)} & & & & & \\ & \phi_{(2)} & & & & \\ & & \phi_{(3)} & & & \\ & & & \sqrt{\phi_{(1)}\phi_{(2)}} & & \\ & & & & \sqrt{\phi_{(2)}\phi_{(3)}} & \\ & & & & & \sqrt{\phi_{(3)}\phi_{(1)}} \end{bmatrix}. \tag{62}$$

$$C_{ijkl} = w_{ip}w_{jq}w_{kr}w_{ls}C^0_{pqrs}, \tag{63}$$

$$C_{ijkl} = \frac{-\nu^0}{E^0}\phi_{ij}\phi_{kl} + \frac{1+\nu^0}{2E^0}\left(\phi_{ik}\phi_{jl} + \phi_{il}\phi_{jk}\right), \tag{64}$$

$$\mathbf{C} = \begin{bmatrix} \phi^2_{(1)}\frac{1}{E^0} & \phi_{(1)}\phi_{(2)}\frac{-\nu^0}{E^0} & \phi_{(1)}\phi_{(3)}\frac{-\nu^0}{E^0} & & & \\ \phi_{(2)}\phi_{(1)}\frac{-\nu^0}{E^0} & \phi^2_{(2)}\frac{1}{E^0} & \phi_{(2)}\phi_{(3)}\frac{-\nu^0}{E^0} & & & \\ \phi_{(3)}\phi_{(1)}\frac{-\nu^0}{E^0} & \phi_{(3)}\phi_{(2)}\frac{-\nu^0}{E^0} & \phi^2_{(3)}\frac{1}{E^0} & & & \\ & & & \phi_{(1)}\phi_{(2)}\frac{2(1+\nu^0)}{E^0} & & \\ & & & & \phi_{(2)}\phi_{(3)}\frac{2(1+\nu^0)}{E^0} & \\ & & & & & \phi_{(3)}\phi_{(1)}\frac{2(1+\nu^0)}{E^0} \end{bmatrix}. \tag{65}$$

A comparison of this matrix with the compliance matrix for orthotropic elasticity (58) yields the same twelve equivalences for the elastic coefficients (59), provided the tensors ϕ_{ij} and $\bar{\phi}_{ij}$ are inverse to each other. Therefore, we verify that both stress and strain formulations based on the product symmetrization of effective quantities are fully equivalent, i.e. $\boldsymbol{\alpha}$ and $\mathbf{C}$ are the inverses of $\bar{\boldsymbol{\alpha}}$ and $\mathbf{E}$.

The same comparison also gives information about the type of anisotropy that is obtained with this approach. The expressions obtained for secant stiffness and compliance contain five independent parameters (E^0, ν^0 plus three principal values of damage), while general orthotropic elasticity has nine. This means that the 'basic' anisotropic damage formulation represents only a relatively restricted form of anisotropy. This is one of the motivations for a more general approach developed in (Carol et al., 1999).

5. Pseudo-logarithmic damage rate and conjugate forces

After the second-order tensor damage variables have been selected, and the dependency of C_{ijkl} (or E_{ijkl}) on those variables established, the next step is to identify the corresponding terms of the general theory described in Section 2. As the general theory was presented in terms of stress and compliance, derivations in this section will also follow the same approach. Among the various tensors defined, the inverse integrity tensor is selected as the primary damage variable $\mathscr{D}_*$. The choice of C_{ijkl} and ϕ_{pq} are only for convenience. Other possibilities would produce dual developments and, with the appropriate substitutions and conversions, final equations should be fully equivalent.

As the first step, C_{ijkl} given in Eq. (64) is differentiated:

$$\dot{C}_{ijkl} = \frac{\partial C_{ijkl}}{\partial \phi_{pq}}\dot{\phi}_{pq}, \tag{66a}$$

$$\frac{\partial C_{ijkl}}{\partial \phi_{pq}} = \frac{-\nu^0}{2E^0}\left[(\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp})\phi_{kl}+\phi_{ij}(\delta_{kp}\delta_{lq}+\delta_{kq}\delta_{lp})\right]+\frac{1+\nu^0}{4E^0}\left[(\delta_{ip}\delta_{kq}+\delta_{iq}\delta_{kp})\phi_{jl}\right.$$
$$\left.+\phi_{ik}(\delta_{jp}\delta_{lq}+\delta_{jq}\delta_{lp})+(\delta_{ip}\delta_{lq}+\delta_{iq}\delta_{lp})\phi_{jk}+\phi_{il}(\delta_{jp}\delta_{kq}+\delta_{jq}\delta_{kp})\right]. \tag{66b}$$

Replacing $\dot{C}_{ijkl}$ into Eq. (13c), one obtains the expression of the dissipation rate from which the conjugate force $-\mathscr{Y}^{\phi}_{pq}$ conjugate to ϕ_{pq} is identified:

$$\dot{d} = (-\mathscr{Y}^{\phi}_{pq})\dot{\phi}_{pq}, \tag{67a}$$

$$-\mathscr{Y}^{\phi}_{pq} = \frac{1}{2}\sigma_{ij}\frac{\partial C_{ijkl}}{\partial \phi_{pq}}\sigma_{kl} = \frac{-\nu^0}{E^0}(\sigma_{kl}\phi_{kl})\sigma_{pq}+\frac{1+\nu^0}{E^0}\sigma_{pk}\phi_{kl}\sigma_{lq}. \tag{67b, c}$$

This force, analogous to what was obtained in earlier papers in terms of stiffness and strains (e.g. Neilsen and Schreyer, 1992), has no clear physical meaning, which makes it difficult to propose and interpret loading functions and damage rules (Chaboche, 1992). From Eq. (50), however, one can see that $\sigma_{kl}\phi_{kl} = \sigma^{\text{eff}}_{kk}$ in Eq. (67c), which motivates the search for an expression of $-\mathscr{Y}_{pq}$ in terms of effective quantities exclusively. This is possible and conveniently achieved by changing the damage variable involved in the dissipation Eq. (67a), from the rate of ϕ_{pq} to the rate of a *pseudo-logarithmic damage* tensor $\dot{L}_{rs}$ defined as

$$\dot{L}_{rs} = 2\bar{w}_{rp}\dot{\phi}_{pq}\bar{w}_{qs} \quad \text{or} \quad \dot{\phi}_{pq} = \tfrac{1}{2}w_{pr}\dot{L}_{rs}w_{sq}. \tag{68a, b}$$

If the principal axes of damage remain constant, the new tensor coincides with the logarithm of the square inverse integrity tensor, i.e. $\mathbf{L} = \ln \boldsymbol{\phi}^2$ (logarithm of a tensor defined as a tensor function, i.e. with same principal axes and logarithm of the principal values). In the case that the principal axes rotate, $\dot{\mathbf{L}}$ defined by Eq. (68a) cannot be guaranteed to be an exact differential, and therefore a general explicit relation between the two tensors is not available. Nevertheless, convenient relations may be established between some of their components and invariants, as developed in Appendix A and used later in the article.

The lack of a general explicit relation between L_{ij} and ϕ_{ij} does not actually represent a practical difficulty because the pseudo-log damage is only used in rate form due to its properties of exhibiting a convenient conjugate force. Once the damage rule in terms of $\dot{L}_{ij}$ is established, the rate $\dot{\phi}_{ij}$ can be always evaluated with Eq. (68b) and the integration process needed in the numerical implementation of the model can be always carried out directly in terms of ϕ_{ij}, which is the variable that enters directly the expressions of secant compliance or stiffness.

With the new pseudo-log damage rate, Eqs. (66a) and (66b) become:

$$\dot{C}_{ijkl} = \frac{\partial C_{ijkl}}{\partial L_{rs}}\dot{L}_{rs}, \tag{69a}$$

$$\frac{\partial C_{ijkl}}{\partial L_{rs}} = \frac{-\nu^0}{4E^0}\left[(w_{ir}w_{js}+w_{is}w_{jr})\phi_{kl}+\phi_{ij}(w_{kr}w_{ls}+w_{ks}w_{lr})\right]+\frac{1+\nu^0}{8E^0}\left[(w_{ir}w_{ks}+w_{is}w_{kr})\phi_{jl}\right.$$
$$\left.+\phi_{ik}(w_{jr}w_{ls}+w_{js}w_{lr})+(w_{ir}w_{ls}+w_{is}w_{lr})\phi_{jk}+\phi_{il}(w_{jr}w_{ks}+w_{js}w_{kr})\right]. \tag{69b}$$

Introducing $\dot{\mathbf{C}}$ into Eq. (13), the dissipation rate is obtained and the new thermodynamic force $-\mathscr{Y}_{rs}$, conjugate to L_{rs}, is identified:

$$\dot{d} = (-\mathscr{Y}_{rs})\dot{L}_{rs}, \qquad -\mathscr{Y}_{rs} = \frac{-\nu^0}{2E^0}\left(\sigma^{\text{eff}}_{kk}\right)\sigma^{\text{eff}}_{rs}+\frac{1+\nu^0}{2E^0}\sigma^{\text{eff}}_{rk}\sigma^{\text{eff}}_{ks}. \tag{70a, b}$$

As linear isotropic elasticity has been assumed as the relation between σ^{eff}_{ij} and $\epsilon^{\text{eff}}_{kl}$, i.e. Eq. (36) with Eq. (23), it is not difficult to verify that this force may be simply rewritten as

$$-\mathscr{Y}_{rs} = \tfrac{1}{2}\sigma^{\text{eff}}_{rk}\epsilon^{\text{eff}}_{ks}. \tag{71}$$

Due to the coaxiality of σ^{eff}_{ij} and $\epsilon^{\text{eff}}_{ij}$, this conjugate force is another symmetric second-order tensor with the same principal axes, and principal values equal to

$$-\mathscr{Y}_{(i)} = \tfrac{1}{2}\sigma^{\text{eff}}_{(i)}\epsilon^{\text{eff}}_{(i)}, \quad i = 1, 3. \tag{72}$$

The conjugate force tensor (71) exhibits the convenient property that its first invariant is equal to the current value of elastic energy

$$-\mathscr{Y}_{rr} = \tfrac{1}{2}\sigma^{\text{eff}}_{rk}\epsilon^{\text{eff}}_{kr} = u, \tag{73}$$

and its volumetric and deviatoric components can be simply expressed as

$$-\mathscr{Y}^{V} = \frac{u}{3}, \qquad -\mathscr{Y}^{D}_{rs} = \frac{1}{2}\sigma^{\text{eff}}_{rk}\epsilon^{\text{eff}}_{ks} - \frac{u}{3}\delta_{rs}. \tag{74a, b}$$

The dissipation rate (70) can be rewritten as the sum of volumetric and deviatoric contributions:

$$\dot{d} = u\dot{L}^{V} + \left(\frac{1}{2}\sigma^{\text{eff}}_{rk}\epsilon^{\text{eff}}_{ks} - \frac{u}{3}\delta_{rs}\right)\dot{L}^{D}_{rs}. \tag{75}$$

This expression may now be compared to the dissipation rate (30) of the isotropic formulation of Section 3. The logarithmic damage rate and conjugate force of that section, both scalar, may be immediately identified with the volumetric components here. This confirms that the basic isotropic formulation is recovered as a particular case when the quasi-log rate of damage stays purely volumetric, i.e. when the damage increments remain isotropic.

6. Loading function and pseudo-log damage rule

It seems natural to define the loading function F in terms of the conjugate forces $-\mathscr{Y}_{ij}$ and of the previous history. Here, we consider the following type of expression:

$$F = f(-\mathscr{Y}_{ij}) - r(\text{history}) = 0. \tag{76}$$

The gradients of the loading function in the damage, compliance and stress spaces may be calculated using Eqs. (19b) and (15b) with $\mathscr{D}_* = L_{ij}$:

$$\mathscr{N}_{rs} = \frac{\partial f}{\partial(-\mathscr{Y}_{rs})}, \qquad N_{ijkl} = \frac{\partial C_{ijkl}}{\partial L_{rs}}\mathscr{N}_{rs}, \qquad n_{ij} = \frac{\partial C_{ijkl}}{\partial L_{rs}}\mathscr{N}_{rs}\sigma_{kl}, \tag{77a, b, c}$$

where $-\mathscr{Y}_{rs}$ is given by Eq. (71) and $\partial C_{ijkl}/\partial L_{rs}$ by Eq. (69b). If the model is associated, $\mathscr{M}_{rs} = \mathscr{N}_{rs}$, $M_{ijkl} = N_{ijkl}$ and $m_{ij} = n_{ij}$. Otherwise, $\mathscr{M}_{rs}$ has to be specified independently, and M_{ijkl} and m_{ij} follow from expressions analogous to Eqs. (77b,c), i.e.

$$\mathscr{M}_{rs} = \frac{\partial g}{\partial(-\mathscr{Y}_{rs})}, \qquad M_{ijkl} = \frac{\partial C_{ijkl}}{\partial L_{rs}}\mathscr{M}_{rs}, \qquad m_{ij} = \frac{\partial C_{ijkl}}{\partial L_{rs}}\mathscr{M}_{rs}\sigma_{kl}. \tag{78a, b, c}$$

Applying the chain rule to the derivatives of F, Eq. (76), one also obtains the hardening/softening parameter

$$H = -\left.\frac{\partial F}{\partial \lambda}\right|_{\sigma} = \frac{\partial r}{\partial \lambda} - \mathscr{N}_{pq}\left.\frac{\partial(-\mathscr{Y}_{pq})}{\partial L_{rs}}\right|_{\sigma}\mathscr{M}_{rs}. \tag{79}$$

From this, we may also obtain $\bar{H} = H + \mathbf{n} : \mathbf{E} : \mathbf{m}$ or, using the dual strain-based derivation (Carol et al., 1994), the more convenient form,

$$\bar{H} = -\frac{\partial F}{\partial \lambda}\bigg|_{\epsilon} = \frac{\partial r}{\partial \lambda} - \mathscr{N}_{pq}\frac{\partial(-\mathscr{Y}_{pq})}{\partial L_{rs}}\bigg|_{\epsilon}\mathscr{M}_{rs}. \tag{80}$$

To formulate the resistance function r, previous history is naturally represented by the current damage state. Thus, we can usually also replace $\partial r/\partial\lambda = (\partial r/\partial L_{rs})\mathscr{M}_{rs}$. With $\mathbf{n}$, $\mathbf{m}$ and $\bar{H}$ obtained from f, g and r, plastic multiplier and tangential stiffness follow automatically using Eqs. (7) and (8), and the formulation is completed. Partial derivatives $\partial(-\mathscr{Y}_{pq})/\partial L_{rs}$ at both constant nominal stress and at constant nominal strain turn out to be lengthy expressions, and are developed in Appendix C.

A simple choice for f is in terms of the invariants of $-\mathscr{Y}_{ij}$. This actually does not contradict the anisotropic nature of the model because in the $-\mathscr{Y}_{ij}$ space, only effective stress and effective strain are involved; if these are replaced using Eqs. (52) and (60b), the damage tensor comes into the picture resulting in an anisotropic loading function in terms of nominal stress or strain. Thus, it makes sense to consider the space of principal values of the conjugate force $-\mathscr{Y}_{(1)}$, $-\mathscr{Y}_{(2)}$, $-\mathscr{Y}_{(3)}$. In that space, one may represent concepts such as p-axis, deviatoric planes, loading surface $F = 0$ and damage rule, analogous to what is customary in the principal stress space in the context of plasticity theory (Fig. 3).

The choice of a pseudo-log damage rate and the space defined by its conjugate force brings about a number of interesting advantages. As shown in Appendix A, it turns out that the *volumetric part* of the pseudo-log flow rule, represented in the $-\mathscr{Y}_{(1)}$, $-\mathscr{Y}_{(2)}$, $-\mathscr{Y}_{(3)}$ space by its component parallel to the p-axis, causes only increments of *isotropic degradation*. On the other hand, the *deviatoric part* of the pseudo-log damage rule (i.e. its component on the deviatoric plane) causes only increments of *anisotropic* degradation. In this way, we have a very simple and understandable separation of effects that may be very useful for the development of specific models (Fig. 3). For instance, it is trivial to verify that the traditional "$(1 - D)$" associated scalar damage model is recovered with a loading surface parallel to the π-plane.

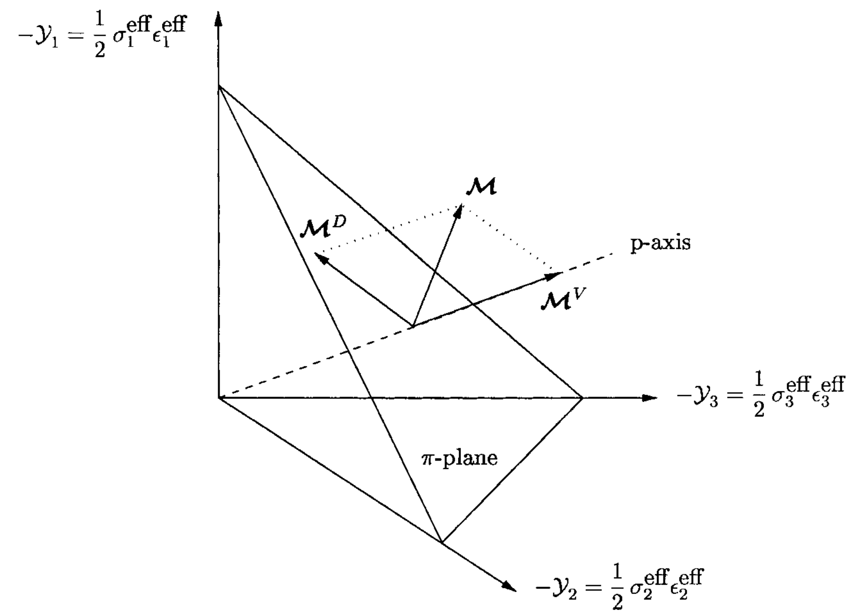

Fig. 3. Space of principal conjugate forces and volumetric/deviatoric (isotropic/anisotropic) decomposition of the damage rule in pseudo-log space.

The condition that the dissipation (70a) must be always positive, leads to the conditions that the loading surface must be convex in the $-\mathscr{Y}_{ij}$ space and must include its origin, analogous to similar arguments classical in elasto-plasticity (Hill, 1950; Malvern, 1969). Additional constraints to the pseudo-log damage rule may be derived from its own definition as the rate of a damage measure. The inverse integrity $\boldsymbol{\phi}$ has been defined as a tensor which evolves between δ_{ij} and ∞ as damage progresses. If s_i denotes the cartesian components of a generic unit vector ($s_i s_i = 1$), the projection $s_i \phi_{ij} s_j$ may be interpreted as a geometric measure of the damage on a plane with normal oriented with that direction (i.e. inverse of a stress-carrying area fraction). Due to the irreversible nature of damage (no healing is considered in this study), it seems reasonable to assume that the damage on any given plane should always increase or remain constant, but never decrease. This means that, for any orientation **s**, we must have

$$s_i \dot{\phi}_{ij} s_j \geqslant 0. \tag{81}$$

This implies that the damage rate tensor $\dot{\phi}_{ij}$ be positive semi-definite, i.e. that all its eigenvalues be positive or zero. By replacing Eq. (68b) in the previous equation, one obtains

$$\frac{1}{2} s'_p \dot{L}_{pq} s'_q = \frac{\dot{\lambda}}{2} s'_p \mathscr{M}_{pq} s'_p \geqslant 0, \quad s'_p = w_{pi} s_i \tag{82}$$

in which the square root integrity tensor w_{pi} is in general non-singular and therefore s'_p is also any arbitrary orientation. Since the inelastic multiplier $\dot{\lambda}$ must be non-negative, this means that the pseudo-log damage rule $\mathscr{M}_{pq}$ must also be positive semi-definite, i.e. that its principal values must satisfy $\mathscr{M}_{(1)} \geqslant 0$, $\mathscr{M}_{(2)} \geqslant 0$ and $\mathscr{M}_{(3)} \geqslant 0$. In terms of a geometric representation in Fig. 3, this implies that the vector representing the damage rule should be part of the positive–positive–positive octant, which is a severe restriction if compared with traditional flow rules in stress space. For instance, associated models with surfaces similar to von Mises or Drucker–Prager (in which the normal may have negative component on one of the axes) are not allowed here. On the other hand, a surface similar to Rankine in the $-\mathscr{Y}_{(1)}$, $-\mathscr{Y}_{(2)}$, $-\mathscr{Y}_{(3)}$ space would sit in the limit of the stated restriction, with only one positive principal value of $\mathscr{M}_{ij}$ at a time, while the other two are zero. This model, that we will call 'pseudo-Rankine', actually exhibits very appealing properties and is developed in detail and illustrated with some application examples in a companion article.

7. Concluding remarks

(1) The theoretical framework for elastic degradation and damage proposed previously by the authors proves to be a powerful and robust tool for developing consistent material formulations in an orderly and systematic manner.

(2) A basic formulation of anisotropic damage based on a second-order damage tensor, energy equivalence and product-type symmetrization, is developed within that framework, and agrees with secant fomulations widely used in the literature. With the usual damage tensors, however, the corresponding conjugate forces are complicated and physically meaningless, which makes it difficult to establish evolution laws.

(3) This problem is overcome by introducing the pseudo-logarithmic rate of damage. This new proposal brings attractive properties to the new damage variable and its corresponding conjugate force, which becomes a simple product of effective stresses and effective strains.

(4) Perhaps the most salient of these properties is the separation of isotropic and anisotropic effects that is induced in pseudo-log space; the volumetric part of the damage rule only generates increments of isotropic damage, while its deviatoric part only generates anisotropic degradation. This separation makes it more physical and intuitive to formulate evolution laws with specific features. The classical isotropic scalar

damage model becomes the simplest first-invariant loading surface represented by cut-off π-planes in the conjugate force space.

(5) A specific implementation of these principles in a 'generalized pseudo-Rankine model' formulation, together with some examples of verification, are included in a companion Part II article.

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Appendix A. Pseudo-logarithmic rate of a symmetric second-order tensor

If $\boldsymbol{\phi}$ is a second-order symmetric tensor with cartesian components ϕ_{ij}, we define its pseudo-logarithmic rate $\dot{L}_{ij}$ as

$$\dot{L}_{ij} = 2\bar{w}_{ip}\dot{\phi}_{pq}\bar{w}_{qj} \tag{A.1}$$

with the notation $\bar{\boldsymbol{\phi}} = \boldsymbol{\phi}^{-1}$ and $\bar{\mathbf{w}}$ equal to the square root tensor of $\bar{\boldsymbol{\phi}}$, so that $\bar{w}_{rp}\bar{w}_{ps} = \bar{\phi}_{rs}$.

The original tensor $\boldsymbol{\phi}$, its inverse and their square roots may be decomposed in product form

$$\phi_{ij} = \phi\psi_{ij}, \quad \bar{\phi}_{ij} = \bar{\phi}\bar{\psi}_{ij}, \quad w_{ij} = wv_{ij}, \quad \bar{w}_{ij} = \bar{w}\bar{v}_{ij}, \tag{A.2a, b, c, d}$$

where

$$\phi = (\det \boldsymbol{\phi})^{1/3}, \quad \bar{\phi} = (\det \bar{\boldsymbol{\phi}})^{1/3} = \frac{1}{\phi}, \quad w = (\det \mathbf{w})^{1/3} = \sqrt{\phi}, \quad \bar{w} = (\det \bar{\mathbf{w}})^{1/3} = \frac{1}{\sqrt{\phi}}, \tag{A.3}$$

$$\psi_{ij} = \frac{1}{\phi}\phi_{ij}, \quad \bar{\psi}_{ij} = \frac{1}{\bar{\phi}}\bar{\phi}_{ij}, \quad v_{ij} = \frac{1}{w}w_{ij}, \quad \bar{v}_{ij} = \frac{1}{\bar{w}}\bar{w}_{ij} \tag{A.4a, b, c, d}$$

in which $\boldsymbol{\psi}$, $\bar{\boldsymbol{\psi}}$, $\mathbf{v}$ and $\bar{\mathbf{v}}$ are all isochoric tensors (namely tensors with unit determinant). These tensors also satisfy $\boldsymbol{\psi} = \mathbf{v}^2$ and $\bar{\boldsymbol{\psi}} = \bar{\mathbf{v}}^2$. Using decomposition (A.2a), the rate of ϕ_{ij} can be expressed as

$$\dot{\phi}_{ij} = \dot{\phi}\psi_{ij} + \phi\dot{\psi}_{ij}. \tag{A.5}$$

If this equation and Eq. (A.2d) are substituted in the pseudo-log rate (A.1), one obtains the additive decomposition:

$$\dot{L}_{ij} = \dot{L}\delta_{ij} + \dot{\mu}_{ij}, \qquad \dot{L} = 2\frac{\dot{\phi}}{\phi}, \qquad \dot{\mu}_{ij} = 2\bar{v}_{ip}\dot{\psi}_{pq}\bar{v}_{qj}, \tag{A.6}$$

where the first component, $\dot{L}\delta_{ij}$, is purely volumetric, and the second, $\dot{\mu}_{ij}$, is purely deviatoric. The latter can be verified by calculating the trace of $\dot{\mu}_{ij}$:

$$\dot{\mu}_{ii} = 2\bar{v}_{ip}\dot{\psi}_{pq}\bar{v}_{qi} = 2\bar{\psi}_{pq}\dot{\psi}_{pq}. \tag{A.7}$$

The tensors $\bar{\psi}_{pq}$ and $\dot{\psi}_{pq}$ can be expressed in terms of the principal values $\psi_{(i)}$ and directions $n_k^{(i)}$ of $\boldsymbol{\psi}$:

$$\dot{\mu}_{ii} = 2\left[\sum_{k=1}^{3} \frac{1}{\psi_{(k)}} n_p^{(k)} n_q^{(k)}\right]\left[\sum_{l=1}^{3} \dot{\psi}_{(l)} n_p^{(l)} n_q^{(l)} + \sum_{l=1}^{3} \psi_{(l)}\left(\dot{n}_p^{(l)} n_q^{(l)} + n_p^{(l)} \dot{n}_q^{(l)}\right)\right]. \tag{A.8}$$

Taking into account that $n_p^{(k)} n_p^{(l)} = \delta_{kl}$ and $\dot{n}_p^{(k)} n_p^{(k)} = 0$, the previous expression can be developed and simplified, and the result multiplied by the unit determinant $\psi_{(1)}\psi_{(2)}\psi_{(3)} = 1$, yielding

$$\dot{\mu}_{ii} = 2\left[\frac{\dot{\psi}_{(1)}}{\psi_{(1)}} + \frac{\dot{\psi}_{(2)}}{\psi_{(2)}} + \frac{\dot{\psi}_{(3)}}{\psi_{(3)}}\right] = 2\left[\dot{\psi}_{(1)}\psi_{(2)}\psi_{(3)} + \psi_{(1)}\dot{\psi}_{(2)}\psi_{(3)} + \psi_{(1)}\psi_{(2)}\dot{\psi}_{(3)}\right] = 0. \tag{A.9}$$

The last expression is equal to zero because is equal to the differentiation of the same unit determinant $\psi_{(1)}\psi_{(2)}\psi_{(3)} = 1$.

The volumetric part of the pseudo-log tensor, $L = L_{kk}/3$, may be explicitly integrated in terms of $\phi = (\det \boldsymbol{\phi})^{1/3}$. Directly from (A.6b), one obtains

$$L = \ln\left(\phi^2\right), \quad \phi = \mathrm{e}^{L/2}. \tag{A.10a, b}$$

Appendix B. Secant stiffness/compliance based on sum-type symmetrization of effective stresses

The fundamental assumption of the sum-type symmetrization may be expressed as (Lemaitre and Chaboche, 1990; Murakami, 1988)

$$\sigma_{ij} = \tfrac{1}{2}\left(\bar{\phi}_{ik}\sigma_{kj}^{\mathrm{eff}} + \sigma_{ik}^{\mathrm{eff}}\bar{\phi}_{kj}\right), \tag{B.1}$$

where $\bar{\phi}_{ij}$ is the integrity tensor. This may be rewritten in terms of the damage–effect tensor $\bar{\boldsymbol{\alpha}}$:

$$\sigma_{ij} = \bar{\alpha}_{ijkl}\sigma_{kl}^{\mathrm{eff}}, \qquad \bar{\alpha}_{ijkl} = \tfrac{1}{4}\left(\bar{\phi}_{ik}\delta_{jl} + \bar{\phi}_{il}\delta_{jk} + \delta_{ik}\bar{\phi}_{jl} + \delta_{il}\bar{\phi}_{jk}\right), \tag{B.2a, b}$$

where advantage has been taken of the symmetry of $\sigma_{kl}^{\mathrm{eff}}$ to obtain an $\bar{\alpha}_{ijkl}$ with all minor and major symmetries. The 6×6 matrix representation of $\bar{\alpha}_{ijkl}$ in the reference system given by the principal directions of damage is diagonal:

$$\bar{\boldsymbol{\alpha}} = \begin{bmatrix} \bar{\phi}_{(1)} & & & & & \\ & \bar{\phi}_{(2)} & & & & \\ & & \bar{\phi}_{(3)} & & & \\ & & & \frac{\bar{\phi}_{(1)}+\bar{\phi}_{(2)}}{2} & & \\ & & & & \frac{\bar{\phi}_{(2)}+\bar{\phi}_{(3)}}{2} & \\ & & & & & \frac{\bar{\phi}_{(3)}+\bar{\phi}_{(1)}}{2} \end{bmatrix}. \tag{B.3}$$

Now substituting Eq. (B.2b) into Eq. (40a) and taking advantage of the symmetry of Kronecker delta δ_{ij} and integrity tensor $\bar{\phi}_{ij}$, one obtains the secant stiffness in terms of the integrity tensor:

$$E_{ijkl} = \Lambda^0\bar{\phi}_{ij}\bar{\phi}_{kl} + \frac{G^0}{2}\left(\bar{\phi}_{ik}\bar{\phi}_{jl} + \bar{\phi}_{il}\bar{\phi}_{jk} + \frac{1}{2}\bar{\phi}_{is}\bar{\phi}_{sk}\delta_{jl} + \frac{1}{2}\bar{\phi}_{is}\bar{\phi}_{sl}\delta_{jk} + \frac{1}{2}\delta_{ik}\bar{\phi}_{js}\bar{\phi}_{sl} + \frac{1}{2}\delta_{il}\bar{\phi}_{js}\bar{\phi}_{sk}\right). \tag{B.4}$$

If compared to its counterpart in the product-type symmetrization (55), this expression differs in the shear terms. The differences may be seen better on the matrix representation of E_{ijkl} in the principal axes of damage:

$$\mathbf{E} = \begin{bmatrix} \bar\phi_{(1)}^2(\Lambda^0+2G^0) & \bar\phi_{(1)}\bar\phi_{(2)}\Lambda^0 & \bar\phi_{(1)}\bar\phi_{(3)}\Lambda^0 & & & \\ \bar\phi_{(2)}\bar\phi_{(1)}\Lambda^0 & \bar\phi_{(2)}^2(\Lambda^0+2G^0) & \bar\phi_{(2)}\bar\phi_{(3)}\Lambda^0 & & & \\ \bar\phi_{(3)}\bar\phi_{(1)}\Lambda^0 & \bar\phi_{(3)}\bar\phi_{(2)}\Lambda^0 & \bar\phi_{(3)}^2(\Lambda^0+2G^0) & & & \\ & & & \left(\frac{\bar\phi_{(1)}+\bar\phi_{(2)}}{2}\right)^2 G^0 & & \\ & & & & \left(\frac{\bar\phi_{(2)}+\bar\phi_{(3)}}{2}\right)^2 G^0 & \\ & & & & & \left(\frac{\bar\phi_{(3)}+\bar\phi_{(1)}}{2}\right)^2 G^0 \end{bmatrix}, \tag{B.5}$$

which is the counterpart of Eq. (56). This stiffness matrix may be inverted with Λ^0 and G^0 being replaced by E^0 and ν^0 to obtain the counterpart of Eq. (57)

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\bar\phi_{(1)}^2E^0} & \frac{-\nu^0}{\bar\phi_{(1)}\bar\phi_{(2)}E^0} & \frac{-\nu^0}{\bar\phi_{(1)}\bar\phi_{(3)}E^0} & & & \\ \frac{-\nu^0}{\bar\phi_{(2)}\bar\phi_{(1)}E^0} & \frac{1}{\bar\phi_{(2)}^2E^0} & \frac{-\nu^0}{\bar\phi_{(2)}\bar\phi_{(3)}E^0} & & & \\ \frac{-\nu^0}{\bar\phi_{(3)}\bar\phi_{(1)}E^0} & \frac{-\nu^0}{\bar\phi_{(3)}\bar\phi_{(2)}E^0} & \frac{1}{\bar\phi_{(3)}^2E^0} & & & \\ & & & \left(\frac{2}{\bar\phi_{(1)}+\bar\phi_{(2)}}\right)^2 \frac{2(1+\nu^0)}{E^0} & & \\ & & & & \left(\frac{2}{\bar\phi_{(2)}+\bar\phi_{(3)}}\right)^2 \frac{2(1+\nu^0)}{E^0} & \\ & & & & & \left(\frac{2}{\bar\phi_{(3)}+\bar\phi_{(1)}}\right)^2 \frac{2(1+\nu^0)}{E^0} \end{bmatrix}. \tag{B.6}$$

This compliance matrix may be compared to the traditional compliance matrix for orthotropic elasticity (58), which results in the following equivalences:

$$\begin{aligned} &E_1 = \bar\phi_{(1)}^2E^0, \quad E_2 = \bar\phi_{(2)}^2E^0, \quad E_3 = \bar\phi_{(3)}^2E^0, \\ &\nu_{12} = \frac{\bar\phi_{(2)}}{\bar\phi_{(1)}}\nu^0, \quad \nu_{13} = \frac{\bar\phi_{(3)}}{\bar\phi_{(1)}}\nu^0, \quad \nu_{21} = \frac{\bar\phi_{(1)}}{\bar\phi_{(3)}}\nu^0, \\ &\nu_{23} = \frac{\bar\phi_{(3)}}{\bar\phi_{(2)}}\nu^0, \quad \nu_{31} = \frac{\bar\phi_{(1)}}{\bar\phi_{(3)}}\nu^0, \quad \nu_{32} = \frac{\bar\phi_{(2)}}{\bar\phi_{(3)}}\nu^0, \\ &G_{12} = \left(\frac{\bar\phi_{(1)}+\bar\phi_{(2)}}{2}\right)^2\frac{E^0}{2(1+\nu^0)}, \quad G_{23} = \left(\frac{\bar\phi_{(2)}+\bar\phi_{(3)}}{2}\right)^2\frac{E^0}{2(1+\nu^0)}, \\ &G_{31} = \left(\frac{\bar\phi_{(3)}+\bar\phi_{(1)}}{2}\right)^2\frac{E^0}{2(1+\nu^0)}. \end{aligned} \tag{B.7}$$

Comparing these equations to their counterparts in the product-type symmetrization (59), Young's moduli and Poisson's ratios are the same, but the shear moduli differ: here the original values G^0 have been reduced by the squares of the sum averages $(\bar\phi_{(i)}+\bar\phi_{(j)})/2$, while there G^0 was affected by the product $\bar\phi_{(i)}\bar\phi_{(j)}$ (i.e. by the square of the product averages of the two quantities).

The sum-type formulation can also be derived in a dual way, i.e. in terms of strain and compliance instead of stress and stiffness. This leads to the following expressions for the dual damage-effect tensor α_{ijkl} and compliance C_{ijkl}:

$$\epsilon_{ij} = \frac{1}{2}\left(\phi_{ik}\epsilon^{\text{eff}}_{kj} + \epsilon^{\text{eff}}_{ik}\phi_{kj}\right), \tag{B.8}$$

$$\epsilon_{ij} = \alpha_{ijkl}\epsilon^{\text{eff}}_{kl}, \qquad \alpha_{ijkl} = \frac{1}{4}\left(\phi_{ik}\delta_{jl} + \phi_{il}\delta_{jk} + \delta_{ik}\phi_{jl} + \delta_{il}\phi_{jk}\right), \tag{B.9}$$

$$C_{ijkl} = \frac{-v^0}{E^0}\phi_{ij}\phi_{kl} + \frac{1+v^0}{4E^0}\left(\phi_{ik}\phi_{jl} + \phi_{il}\phi_{jk} + \frac{1}{2}\phi_{is}\phi_{sk}\delta_{jl} + \frac{1}{2}\phi_{is}\phi_{sl}\delta_{jk} + \frac{1}{2}\delta_{ik}\phi_{js}\phi_{sl} + \frac{1}{2}\delta_{il}\phi_{js}\phi_{sk}\right). \tag{B.10}$$

If the reference system coincides with the principal axes of damage, these fourth-order tensors exhibit the following 6×6 matrix representation:

$$\boldsymbol{\alpha} = \begin{bmatrix} \phi_{(1)} & & & & & \\ & \phi_{(2)} & & & & \\ & & \phi_{(3)} & & & \\ & & & \frac{\phi_{(1)}+\phi_{(2)}}{2} & & \\ & & & & \frac{\phi_{(2)}+\phi_{(3)}}{2} & \\ & & & & & \frac{\phi_{(3)}+\phi_{(1)}}{2} \end{bmatrix}, \tag{B.11}$$

$$\mathbf{C} = \begin{bmatrix} \phi^2_{(1)}\frac{1}{E^0} & \phi_{(1)}\phi_{(2)}\frac{-v^0}{E^0} & \phi_{(1)}\phi_{(3)}\frac{-v^0}{E^0} & & & \\ \phi_{(2)}\phi_{(1)}\frac{-v^0}{E^0} & \phi^2_{(2)}\frac{1}{E^0} & \phi_{(2)}\phi_{(3)}\frac{-v^0}{E^0} & & & \\ \phi_{(1)}\phi_{(3)}\frac{-v^0}{E^0} & \phi_{(2)}\phi_{(3)}\frac{-v^0}{E^0} & \phi^2_{(3)}\frac{1}{E^0} & & & \\ & & & \left(\frac{\phi_{(1)}+\phi_{(2)}}{2}\right)^2\frac{2(1+v^0)}{E^0} & & \\ & & & & \left(\frac{\phi_{(2)}+\phi_{(3)}}{2}\right)^2\frac{2(1+v^0)}{E^0} & \\ & & & & & \left(\frac{\phi_{(3)}+\phi_{(1)}}{2}\right)^2\frac{2(1+v^0)}{E^0} \end{bmatrix}. \tag{B.12}$$

Identification of the various components with the traditional orthotropic elastic compliance (58) yields

$$E_1 = \frac{1}{\phi^2_{(1)}}E^0, \quad E_2 = \frac{1}{\phi^2_{(2)}}E^0, \quad E_3 = \frac{1}{\phi^2_{(3)}}E^0,$$

$$v_{12} = \frac{\phi_{(1)}}{\phi_{(2)}}v^0, \quad v_{13} = \frac{\phi_{(1)}}{\phi_{(3)}}v^0, \quad v_{21} = \frac{\phi_{(2)}}{\phi_{(1)}}v^0,$$

$$v_{23} = \frac{\phi_{(2)}}{\phi_{(3)}}v^0, \quad v_{31} = \frac{\phi_{(3)}}{\phi_{(1)}}v^0, \quad v_{32} = \frac{\phi_{(3)}}{\phi_{(2)}}v^0,$$

$$G_{12}=\left(\frac{2}{\phi_{(1)}+\phi_{(2)}}\right)^2\frac{E^0}{2(1+\nu^0)},\quad G_{23}=\left(\frac{2}{\phi_{(2)}+\phi_{(3)}}\right)^2\frac{E^0}{2(1+\nu^0)},$$
$$G_{31}=\left(\frac{2}{\phi_{(3)}+\phi_{(1)}}\right)^2\frac{E^0}{2(1+\nu^0)}. \tag{B.13}$$

With the assumption that the tensors $\bar{\phi}_{ij}$ and ϕ_{ij} are inverse to each other (i.e. that $\bar{\phi}_{(i)}=1/\phi_{(i)}$, $i=1$–3), the previous expressions are *not* equivalent to their stress-based counterparts (B.3), (B.6) and (B.7). Although Young's moduli and Poisson's ratios coincide in both versions of the theory, the three shear moduli do not, and neither do the shear-related terms of the damage–effect tensors (i.e. the inverse of $\phi_{(1)}+\phi_{(2)}$ is *not* equal to $1/\phi_{(1)}+1/\phi_{(2)}$).

Therefore, it can be concluded that the sum-type symmetrization in terms of stress/stiffness and its dual in terms of stress/compliance are *not* equivalent, and they actually represent alternative incompatible approaches.

Appendix C. Partial derivatives $\partial(-\mathcal{Y}_{pq})/\partial L_{rs}$ at constant nominal stress or constant nominal strain

Considering first the derivative at constant nominal stress, using the chain rule we can decompose the desired derivative in the following product of two terms:

$$\left.\frac{\partial(-\mathcal{Y}_{ij})}{\partial L_{rs}}\right|_\sigma=\left.\frac{\partial(-\mathcal{Y}_{ij})}{\partial w_{ab}}\right|_\sigma\left.\frac{\partial w_{ab}}{\partial L_{rs}}\right|_\sigma. \tag{C.1}$$

To obtain the first term of the product, we start from the conjugate force (71) and replace effective strain in terms of effective stress using the elastic isotropic law (36b) with (23b), and effective stress in terms of nominal stress (60b). This yields

$$-\mathcal{Y}_{ij}=\frac{-\nu^0}{2E^0}\left(w_{pq}\sigma_{qr}w_{rp}\right)w_{ik}\sigma_{kl}w_{lj}+\frac{1+\nu^0}{2E^0}w_{ik}\sigma_{kl}w_{lp}w_{pq}\sigma_{qr}w_{rj}. \tag{C.2}$$

Now, we can make the derivative with respect to the symmetric tensor w_{ab}, at constant nominal stress. By doing so, and after some rearrangements and substitution of nominal stress back in terms of effective stress (50), we obtain

$$\begin{aligned}\left.\frac{\partial(-\mathcal{Y}_{ij})}{\partial w_{ab}}\right|_\sigma=&\frac{-\nu^0}{4E^0}\left[2\left(\sigma^{\text{eff}}_{ak}\bar{w}_{kb}+\bar{w}_{ak}\sigma^{\text{eff}}_{kb}\right)\sigma^{\text{eff}}_{ij}+\left(\sigma^{\text{eff}}_{kk}\right)\left((\delta_{ai}\bar{w}_{bk}+\delta_{bi}\bar{w}_{ak})\sigma^{\text{eff}}_{kj}+(\delta_{aj}\bar{w}_{bk}+\delta_{bj}\bar{w}_{ak})\sigma^{\text{eff}}_{ki}\right)\right]\\&+\frac{1+\nu^0}{4E^0}\left[(\delta_{ai}\bar{w}_{bk}+\delta_{bi}\bar{w}_{ak})\left(\sigma^{\text{eff}}\right)^2_{kj}+\sigma^{\text{eff}}_{ik}\left(\bar{w}_{ka}\sigma^{\text{eff}}_{bj}+\bar{w}_{kb}\sigma^{\text{eff}}_{aj}\right)\right.\\&\left.+(\delta_{aj}\bar{w}_{bk}+\delta_{bj}\bar{w}_{ak})\left(\sigma^{\text{eff}}\right)^2_{ki}+\sigma^{\text{eff}}_{jk}\left(\bar{w}_{ka}\sigma^{\text{eff}}_{bi}+\bar{w}_{kb}\sigma^{\text{eff}}_{ai}\right)\right].\end{aligned} \tag{C.3}$$

The derivative in the second term on the right-hand side of Eq.(C.1) is actually the same at constant stress or strain, because w_{ab} is the square root of the integrity tensor ϕ_{cd} and the rate of this is directly related to the pseudo-log rate of damage $\dot{L}_{ab}$ by (68), without any other variable involved. To calculate this factor, we decompose again this derivative into two using the chain rule:

$$\frac{\partial w_{ab}}{\partial L_{rs}}=\frac{\partial w_{ab}}{\partial\phi_{cd}}\frac{\partial\phi_{cd}}{\partial L_{rs}}. \tag{C.4}$$

The second factor here is the easiest since, directly from the definition of pseudo-log rate (68) and taking into account the symmetry of $\dot{L}_{rs}$, one has

$$\frac{\partial \phi_{cd}}{\partial L_{rs}} = \tfrac{1}{4}(w_{cr} w_{ds} + w_{ds} w_{cr}). \tag{C.5}$$

The first factor in Eq. (C.4) is a little trickier. It represents the derivative of the symmetric second-order tensor $\mathbf{w}$ with respect to its second power $\boldsymbol{\phi} = \mathbf{w}^2$. What can be easily obtained is its inverse, by differentiating the square relation $\phi_{cd} = w_{ca} w_{ad}$:

$$\dot{\phi}_{cd} = w_{ck} \dot{w}_{ad} + \dot{w}_{ca} w_{ad} = \frac{\partial \phi_{cd}}{\partial w_{ab}} \dot{w}_{ab}, \tag{C.6a}$$

$$\frac{\partial \phi_{cd}}{\partial w_{ab}} = \frac{1}{2}(w_{ca}\delta_{db} + w_{cb}\delta_{da} + w_{da}\delta_{cb} + w_{db}\delta_{ca}), \tag{C.6b}$$

where advantage has been taken of the symmetry of $\dot{\mathbf{w}}$. In the principal axes of damage, this fourth-order tensor has a simple 6×6 diagonal matrix representation that can be immediately inverted to obtain the desired derivative in the same axes:

$$\frac{\partial \boldsymbol{\phi}}{\partial \mathbf{w}} = \begin{bmatrix} 2w_{(1)} & & & & & \\ & 2w_{(2)} & & & & \\ & & 2w_{(3)} & & & \\ & & & w_{(1)} + w_{(2)} & & \\ & & & & w_{(2)} + w_{(3)} & \\ & & & & & w_{(3)} + w_{(1)} \end{bmatrix}, \tag{C.7a}$$

$$\frac{\partial \mathbf{w}}{\partial \boldsymbol{\phi}} = \begin{bmatrix} \frac{1}{2w_{(1)}} & & & & & \\ & \frac{1}{2w_{(2)}} & & & & \\ & & \frac{1}{2w_{(3)}} & & & \\ & & & \frac{1}{w_{(1)}+w_{(2)}} & & \\ & & & & \frac{1}{w_{(2)}+w_{(3)}} & \\ & & & & & \frac{1}{w_{(3)}+w_{(1)}} \end{bmatrix}. \tag{C.7b}$$

However, in order to be introduced in previous equations, we need a tensorial expression of this derivative which is not as simple as the matrix form would suggest. Hoger and Carlson (1984) give the general tensorial form, which turns out to be a sum of 18 different products of ϕ_{ij}, w_{ij} and δ_{ij}, each of them multiplied by a scalar factor function of the principal values w_1, w_2 and w_3. To our purposes, the alternative formulas given in of (Ogden, 1984, pp. 162–163) for the derivative of a tensor function of a tensor in terms of its spectral decomposition, seem more advantageous. If the unit vectors $s_i^{(I)}$, $I = 1,2,3$ denote the principal directions of damage, i.e.

$$w_{ij} = \sum_{I=1}^{3} w_{(I)}\, s_i^{(I)} s_j^{(I)}, \qquad \phi_{ij} = \sum_{I=1}^{3} \phi_{(I)}\, s_i^{(I)} s_j^{(I)}, \tag{C.8a, b}$$

where implicit summation does not apply to indices between parentheses, Ogden's equations applied to our case give

$$\frac{\partial w_{ab}}{\partial \phi_{cd}} = \frac{1}{2}\sum_{I=1}^{3}\frac{1}{w_{(I)}}s_a^{(I)}s_b^{(I)}s_c^{(I)}s_d^{(I)} + \frac{1}{2}\sum_{I,J\neq I}^{3}\frac{1}{w_{(I)}+w_{(J)}}s_a^{(I)}s_b^{(J)}\left(s_c^{(I)}s_d^{(J)} + s_c^{(J)}s_d^{(I)}\right). \tag{C.9}$$

Note that for this particular tensor function ($\mathbf{w} = \boldsymbol{\phi}^{1/2}$), it turns out that the terms of the first summatory are equal to the terms that would be obtained, if $I = J$ in the second summatory. This allows one to merge both in the simpler form:

$$\frac{\partial w_{ab}}{\partial \phi_{cd}} = \frac{1}{4}\sum_{I,J=1}^{3}\frac{1}{w_{(I)}+w_{(J)}}\left(s_a^{(I)}s_b^{(J)} + s_a^{(J)}s_b^{(I)}\right)\left(s_c^{(I)}s_d^{(J)} + s_c^{(J)}s_d^{(I)}\right) \tag{C.10}$$

in which major symmetry has also been introduced, motivated by the major symmetry observed for the inverse tensor (C.6b). The verification that, in this symmetrized form, tensors (C.10) and (C.6b) are inverse to each other is immediate by making the product and taking into account that $\sum_{I=1}^{3} s_i^{(I)}s_j^{(I)} = \delta_{ij}$. Also, by reducing the unit vectors $s_i^{(I)}$ to their canonical form it is possible to verify that the matrix form (C.7) is recovered.

Eqs. (C.9) and (C.5) may now be introduced into (C.4). Taking advantage that $s_c^{(I)}w_{cr} = w_{(I)}s_r^{(I)}$, this leads to

$$\begin{aligned}\frac{\partial w_{ab}}{\partial L_{rs}} &= \frac{1}{8}\sum_{I=1}^{3}\sum_{J=1}^{3}\frac{w_{(I)}w_{(J)}}{w_{(I)}+w_{(J)}}\left(s_a^{(I)}s_b^{(J)} + s_a^{(J)}s_b^{(I)}\right)\left(s_r^{(I)}s_s^{(J)} + s_r^{(J)}s_s^{(I)}\right)\\ &= \frac{1}{8}\sum_{I=1}^{3}\sum_{J=1}^{3}\frac{1}{\bar{w}_{(I)}+\bar{w}_{(J)}}\left(s_a^{(I)}s_b^{(J)} + s_a^{(J)}s_b^{(I)}\right)\left(s_r^{(I)}s_s^{(J)} + s_r^{(J)}s_s^{(I)}\right).\end{aligned} \tag{C.11}$$

This and Eq. (C.3) may be finally introduced into Eq. (C.1) to obtain the original partial derivative at constant nominal stress.

The partial derivative at constant strain may be obtained following a dual derivation:

$$\left.\frac{\partial(-\mathscr{Y}_{ij})}{\partial L_{rs}}\right|_{\epsilon} = \left.\frac{\partial(-\mathscr{Y}_{ij})}{\partial \bar{w}_{ab}}\right|_{\epsilon}\frac{\partial \bar{w}_{ab}}{\partial L_{rs}}, \tag{C.12}$$

where

$$\begin{aligned}\left.\frac{\partial(-\mathscr{Y}_{ij})}{\partial \bar{w}_{ab}}\right|_{\epsilon} = {}& \frac{\Lambda^0}{4}\left[2\left(\epsilon_{ak}^{\text{eff}}w_{kb} + w_{ak}\epsilon_{kb}^{\text{eff}}\right)\epsilon_{ij}^{\text{eff}} + \left(\epsilon_{kk}^{\text{eff}}\right)\left(\left(\delta_{ai}w_{bk} + \delta_{bi}w_{ak}\right)\epsilon_{kj}^{\text{eff}} + \left(\delta_{aj}w_{bk} + \delta_{bj}w_{ak}\right)\epsilon_{ki}^{\text{eff}}\right)\right]\\ &+ \frac{G^0}{2}\left[\left(\delta_{ai}w_{bk} + \delta_{bi}w_{ak}\right)\left(\epsilon^{\text{eff}}\right)_{kj}^2 + \epsilon_{ik}^{\text{eff}}\left(w_{ka}\epsilon_{bj}^{\text{eff}} + w_{kb}\epsilon_{aj}^{\text{eff}}\right)\right.\\ &\left. + \left(\delta_{aj}w_{bk} + \delta_{bj}w_{ak}\right)\left(\epsilon^{\text{eff}}\right)_{ki}^2 + \epsilon_{jk}^{\text{eff}}\left(w_{ka}\epsilon_{bi}^{\text{eff}} + w_{kb}\epsilon_{ai}^{\text{eff}}\right)\right],\end{aligned} \tag{C.13}$$

and

$$\frac{\partial \bar{w}_{ab}}{\partial L_{rs}} = -\frac{1}{8}\sum_{I=1}^{3}\sum_{J=1}^{3}\frac{1}{w_{(I)}+w_{(J)}}\left(s_a^{(I)}s_b^{(J)} + s_a^{(J)}s_b^{(I)}\right)\left(s_r^{(I)}s_s^{(J)} + s_r^{(J)}s_s^{(I)}\right). \tag{C.14}$$

References

Betten, J., 1983. Damage tensors in continuum mechanics. J. de Mécanique Théorique et Appliquée 2, 13–32.

Carol, I., Bažant, Z.P., 1997. Damage and plasticity in microplane theory. Int. J. Solids Struct. 34 (29), 3807–3835.

Carol, I., Bažant, Z.P., Prat, P.C., 1991. Geometric damage tensor based on microplane model. ASCE J. Engng. Mech. 117, 2429–2448.

Carol, I., Rizzi, E., William, K., 1994. A unified theory of elastic degradation and damage based on a loading surface. Int. J. Solids Struct. 31 (20), 2835–2865.

Carol, I., Rizzi, E., William, K., 1998. On the formulation of isotropic and anisotropic damage. In: Mang, H., Bićanić, N., de Borst, R. (Eds.), Computational Modelling of Concrete Structures, Balkema, Badgastein, Austria, pp. 183–192.

Carol, I., Rizzi, E., William, K., 1999. An extended formulation of isotropic and anisotropic damage with evolution laws in pseudo-log space. Technical Report CU/SR-99/4, Department of CEAE, University of Colorado, Boulder, CO 80309-0428, USA.

Carol, I., Rizzi, E., William, K., 2000. On the formulation of anisotropic elastic degradation. Part II: Generalized pseudo-Rankine model for tensile damage. Int. J. Solids Struct. 38.

Carol, I., William, K., 1996. Spurious energy dissipation/generation in modeling of stiffness recovery for elastic degradation and damage. Int. J. Solids Struct. 33 (20–22), 2939–2957.

Carol, I., William, K., 1997. Application of analytical solutions in elasto-plasticity to localization analysis of damage models. In: Owen, D.R.J., Oñate, E., Hinton, E. (Eds.), Computational Plasticity COMPLAS V, Pineridge Press, Barcelona, pp. 714–719.

Chaboche, J.L., 1978. Description thermodynamique et phénoménologique de la viscoplasticité cyclique avec endommagement. Ph.D thesis, University of Paris IV, France.

Chaboche, J.L., 1981. Damage mechanics in the post-failure regime. Nucl. Engng. Design 64, 233–247.

Chaboche, J.L., 1990. On the description of damage induced anisotropy and active–passive damage effects. In: Ju, J.W. et al. (Eds.), Damage Mechanics in Engineering Materials, AMD, vol. 109. ASME, New York, pp. 153–166.

Chaboche, J.L., 1992. Damage-induced anisotropy: on the difficulties associated with the active–passive unilateral condition. Int. J. Damage Mech. 1, 148–171.

Chaboche, J.L., 1993. Development of continuum damage mechanics for elastic solids sustaining anisotropic and unilateral condition. Int. J. Damage Mech. 2, 311–329.

Chow, J.L., Wang, J., 1987a. An anisotropic theory of continuum damage mechanics for ductile fracture. Engng. Fract. Mech. 27, 547–558.

Chow, J.L., Wang, J., 1987b. An anisotropic theory of elasticity for continuum damage mechanics. Int. J. Fract. 33, 3–16.

Cordebois, J.P., Sidoroff, F., 1982. Endommagement anisotrope en élasticité et plasticité. J. de Mécanique Théorique et Appliquée, Numéro Spécial, pp. 45–60.

Costin, L.S., 1985. Damage mechanics in the post-failure regime. Mech. Mater. 4, 149–160.

Davidson, L., Stevens, A.L., 1973. Thermomechanical constitution of spalling elastic bodies. J. Appl. Phys. 44, 667–674.

Dougill, J.W., 1976. On stable progressively fracturing solids. J. Appl. Math. Phys. 27, 423–437.

Dragon, A., Mróz, Z., 1979. A continuum model for plastic-brittle behaviour of rock and concrete. Int. J. Engng. Sci. 17, 121–137.

Franziskonis, G., Desai, C.S., 1987. Constitutive model with strain softening. Int. J. Solids Struct. 23, 733–750.

Han, D.J., Chen, W.F., 1986. Strain-space plasticity formulation for hardening–softening materials with elasto-plastic coupling. Int. J. Solids Struct. 22, 935–950.

Hansen, N.R., Schreyer, H.L., 1992. Thermodynamically consistent theories of elastoplasticity coupled with damage. In: Ju, J.W., Valanis, K.C. (Eds.), Damage Mechanics and Localization, AMD vol. 142, ASME, New York, pp. 53–67.

Hill, R., 1950. Mathematical Theory of Plasticity. Clarendon Press, Oxford.

Hoger, A., Carlson, D.E., 1984. On the derivative of the square root of a tensor and Guo's rate theorems. J. Elasticity 14, 329–336.

Hueckel, T., Maier, G., 1977. Incrementally boundary value problems in the presence of coupling of elastic and plastic deformations: a rock mechanics oriented theory. Int. J. Solids Struct. 13, 1–15.

Ju, J.W., 1989. On energy-based coupled elastoplastic damage theories. Int. J. Solids Struct. 25, 803–833.

Ju, J.W., 1990. Isotropic and anisotropic damage variables in continuum damage mechanics. ASCE J. Engng. Mech. 116 (12), 2764–2770.

Kachanov, L.M., 1958. Time rupture process under creep conditions. Izv. A Rad. Nauk. SSSR otd Tekh. Nauk. 8, 26–31, in Russian.

Kachanov, M., 1980. Continuum model of medium with cracks. ASCE J. Engng. Mech. Div. 106 (EM5), 1039–1051.

Krajcinovic, D., Fonseka, G.U., 1981. The continuous damage theory of brittle materials. J. Appl. Mech. 48, 809–824.

La Borderie, C., Berthaud, I., Pijaudier-Cabot, G., 1990. Crack closure effects in continuum damage mechanics, numerical implementation. In: Bićanić, N., Mang, H. (Eds.), Computer-Aided Analysis and Design of Concrete Structures, Pineridge Press, Zell-am-See, Austria, pp. 975–986.

Ladevèze, P., 1983. Sur une théorie de l'endommagement anisotrope. Technical Report 34, Laboratoire de Mechanique et Technologie, Ecole Normale Superieure, 61, av. du Pdt. Wilson, 94235 Cachan Cedex, France.

Leckie, F.A., 1978. The constitutive equations of continuum creep damage mechanics. Phil. Trans. Roy. Soc. Lond., 288, 27–47.

Leckie, F.A., Onat, E.T., 1981. Tensorial nature of damage measuring internal variables. In: Hult, J., Lemaitre, J.L. (Eds.), Proceedings of IUTAM Symposium on Physical Nonlinearities in Structural Analysis, Springer, Senlis, France. pp. 140–155.
Lemaitre, J., Chaboche, J.L., 1978. Aspect phenomenologique de la rupture par endommagement. J. de Mécanique Théorique et Appliquée 2, 317–365.
Lemaitre, J., Chaboche, J.L., 1990. Mechanics of Solid Materials. Cambridge University Press, Cambridge. Trans.: Dunod, Paris, 1985 (in French).
Lubarda, V.A., Krajcinovic, D., 1993. Damage tensors and the crack density distribution. Int. J. Solids Struct. 30, 2859–2877.
Lubarda, V.A., Krajcinovic, D., Mastilovic, S., 1994. Damage model for brittle elastic solids with unequal tensile and compressive strengths. Engng. Fract. Mech. 49 (5), 681–697.
Maier, G., Hueckel, T., 1979. Nonassociated and coupled flow rules of elastoplasticity for rock-like materials. Int. J. Rock Mech. Min. Sci. Geomech. Abstr. 16, 77–92.
Malvern, L.E., 1969. Introduction to the Mechanics of a Continuum Medium. Prentice-Hall, NJ.
Mazars, J., Lemaitre, J., 1984. Application of continuous damage mechanics to strain and fracture behavior of concrete. In: Shah, S.P. (Ed.), Application of Fracture Mechanics to Cementitious Composites. Evanston, IL, pp. 375–378.
Mazars, J., Pijaudier-Cabot, G., 1989. Continuum damage theory – Application to concrete. ASCE J. Engng. Mech. 115, 345–365.
Murakami, S., 1987. Anisotropic damage theory and its application to creep crack growth analysis. In: Desai, C.S., Krempl, E., Kiousis, P.D., Kundu, T. (Eds.), Constitutive Laws for Engineering Materials: Theory and Applications, Elsevier, Tucson, USA, pp. 187–194.
Murakami, S., 1988. Mechanical modeling of material damage. J. Appl. Mech. 55, 280–286.
Murakami, S., Ohno, N., 1980. A continuum theory of creep and creep damage. In: Ponter, A.R.S., et al. (Eds.), Creep in Structures, Springer, Berlin, pp. 422–444.
Neilsen, M.K., Schreyer, H.L., 1992. Bifurcations in elastic-damaging materials. In: Ju, J.W., Valanis, K.C. (Eds.), Damage Mechanics and Localization, AMD, vol. 142. ASME, New York, pp. 109–123.
Oda, M., 1983. A method for evaluating the effect of crack geometry on the mechanical behavior of cracked rock masses. Mech. Mater. 2, 163–171.
Ogden, R.W., 1984. Non-Linear Elastic Deformations. Wiley, New York (reprinted by Dover Publications).
Ortiz, M., 1985. A constitutive theory for the inelastic behavior of concrete. Mech. Mater. 4, 67–93.
Resende, L., 1987. A damage mechanics constitutive theory for the inelastic behavior of concrete. Comp. Meth. Appl. Mech. Engng. 60, 57–93.
Rizzi, E., Carol, I., William, K., 1995. Localization analysis of elastic degradation with application to scalar damage. ASCE J. Engng. Mech. 121 (4), 541–554.
Rizzi, E., Maier, G., William, K., 1996. On failure indicators in multi-dissipative materials. Int. J. Solids Struct. 33 (20–22), 3187–3214.
Shen, W., Pen, L.-H., Yue, Y.-G., Shen, Z., Tang, X.-D., 1989. Elastic damage and energy dissipation in anisotropic solid material. Engng. Fract. Mech. 33, 273–281.
Simó, J.C., Ju, J.W., 1987. Stress and strain based continuum damage models, Part I and II. Int. J. Solids Struct. 23, 375–400.
Suaris, W., 1987. A damage theory for concrete incorporating crack growth characteristics. In: Desai, C.S., Krempl, E., Kiousis, P.D., Kundu, T. (Eds.), Constitutive Laws for Engineering Materials, Theory and Applications, vol. 2. Elsevier, Tucson, USA, pp. 931–998.
Suaris, W., Shah, S.P., 1984. Rate-sensitivity damage theory for brittle solids. ASCE J. Engng. Mech. 110 (6), 985–997.
Swoboda, G., Ito, F., 1993. Two-dimensional damage failure propagation model of jointed rock mass. In: Valliappan, S, et al. (Eds.), Computational Mechanics, Balkema, Rotterdam, pp. 681–687.
Vakulenko, A.A., Kachanov, M., 1971. Continuum theory of medium with cracks. Mech. Solids 6, 145–151.
Valanis, K.C., 1990. A theory of damage in brittle materials. Engng. Fract. Mech. 36, 403–416.
Voyiadjis, G.Z., Park, T., 1997. Anisotropic damage-effect tensors for the symmetrization of the effective stress tensor. J. Appl. Mech. 64, 106–110.
Yazdani, S., Schreyer, H.L., 1988. An anisotropic damage model with dilatation for concrete. Mech. Mater. 7, 231–244.